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Pricing of American Path-Dependent Contingent Claims

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Abstract

We consider the problem of pricing path-dependent contingent claims. Classically, this problem can be cast into the Black-Scholes valuation framework through inclusion of the path-dependent variables into the state space. This leads to solving a degenerate advection-diffusion Partial Differential Equation (PDE). Standard Finite Difference (FD) methods are known to be inadequate for solving such degenerate PDE. Hence, path-dependent European claims are typically priced through Monte-Carlo simulation. To date, there is no numerical method for pricing path-dependent American claims.

We first establish necessary and sufficient conditions amenable to a Lie algebraic characterization, under which degenerate diffusions can be reduced to lower-dimensional non-degenerate diffusions on a sub-manifold of the underlying asset space. We apply these results to path-dependent options. Then, we describe a new numerical technique, called *Forward Shooting Grid* (FSG) method, that efficiently copes with degenerate diffusion PDE. Finally, we show that the FSG method is unconditionally stable and convergent.

The FSG method has been implemented for a number of popular path-dependent options, and proved to be much faster than traditional Monte Carlo simulation, for a comparable accuracy. Depending on the type of option, the computation time lies between 1 and 15 seconds on a PC, for a 0.1 % precision.

The FSG method is also the first capable of dealing with the early exercise condition of American options. Furthermore, when the stock price S follows a binomial process, the method computes the *exact* price of any American lookback option on S. The same is true for barrier options, such as up-and-in or down-and-out options.

Several numerical examples are presented and discussed, showing in particular that the Snell envelope upper bounds obtained on American lookback prices are quite overestimated, and also that the usual geometric average approximation to the arithmetic average-rate option is fairly inaccurate.

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Contents

1 Introduction

Path-dependent options are options whose payoffs depend on historical values of the underlying asset over a given time period as well as its current value. Well-known examples are lookback call (put) options, which give their owners the right to buy (sell) the asset at an exercise price equal to the minimum (maximum) price of the asset over the life of the option. Many other variants exist, e.g. capped options, barrier options etc.

Average-rate or Asian options constitute another class of path-dependent instruments, whose payoffs depend on the arithmetic average value of the asset price for some time period. Popular examples include fixed-strike, and floating-strike average-rate options.

Since their introduction in 1982, path-dependent options found their way in several places such as common stocks and foreign exchange markets, by meeting specific risk management and investment needs. As evidence of their increasing popularity, some instruments such as capped options have recently begun to be traded on the Chicago Board of Options Exchange and American Stock Exchange, see e.g. Hunter and Stowe (1992).

The modern approach to path-dependent option pricing relies on the dynamic hedging principle of the Black-Scholes model (Black and Scholes (1973)). In their seminal work, Goldman et al. (1979) have shown that a hedge portfolio could be constructed for an option to buy at the historical maximum, and that closed-form valuation formulas exist in the European case. Garman (1989) developed another valuation model, which separates the lookback option into two underlying options, and gives furthermore the ability to price a European option on an asset that pays dividends. The Asian option is analyzed by Yor (1989). Conze and Viswanathan (1991) derive explicit valuation formulas of most European barrier options, as well as some upper bounds in the American case. Kemna and Vorst (1990) use Monte Carlo simulation with a specific variance reduction method to compute the price of fixed-strike average-rate options.

The path-dependent pricing problem can be cast into the classical Black-Scholes valuation framework through inclusion of the path dependent variables into the state space (see e.g. Stanton (1989)). In a few simple cases, the resulting augmented PDE admits a closed form solution. However, the use of numerical techniques is mandatory in most practical situations. The augmented PDE associated with a path-dependent problem is generally degenerate, i.e. the instantaneous covariance matrix is singular. Finite Difference (FD) methods are numerically stable, but they introduce a spurious additional numerical diffusion, and therefore do not converge towards the theoretical solution. The only practical technique to date consists in computing the price through Monte Carlo simulation by means of the Feynman-Kac representation. This approach is satisfactory for pricing European path-dependent contingent claims. However, it cannot deal with the early exercise conditions of American claims.

In this paper, we first establish necessary and sufficient conditions under which degenerate diffusion PDE can be reduced to lower-dimensional non-degenerate PDE on a sub-manifold of the state space. A degenerate PDE that can be reduced in such a way is called *holonomic*. It is called *non-holonomic* otherwise. It is important to determine whether a given PDE is holonomic or not, since this property influences both the type of numerical integration technique that can be used, and the memory required to conduct the integration.

For this purpose, we use the concept of symmetric multiplication for stochastic integrals, which leads to the notion of Fisk-Stratonovitch differential. Then, we use standard results regarding discrete approximations of multidimensional diffusion processes that establish a link between stochastic differential equations and deterministic optimal control theory. Finally, we use standard results on the accessibility of deterministic control systems. This leads to a Lie algebraic characterization of holonomic PDE.

Then, we apply these results to popular types of path-dependent pricing problems. We show in particular that the average-rate option pricing PDE is non-holonomic. Finally, we present a new numerical technique, called *Forward Shooting Grid* method, that efficiently copes with the degeneracy of these non-holonomic PDE. In particular, we show that, unlike FD methods, the FSG method is unconditionally stable and convergent in the presence of arbitrary degeneracies.

The FSG method has been implemented, and demonstrated the following capabilities:

1) It is much faster than traditional Monte Carlo simulation, for a comparable accuracy. Depending on the type of option, the computation time on a PC lies between 1 and 15 seconds, for a 0.1% precision.

2) It can deal with the early exercise condition of American options. To the best of our knowledge, this is the first method capable of pricing American path-dependent options.

3) When the stock price S follows a Cox-Ross-Rubinstein binomial process, the method computes *exactly* the price of any (e.g. minimum or maximum) *American* lookback option on S. This is also true for barrier options, such as up-and-in or down-and-out options.

The next section briefly reviews the basic ingredients of the modern contingent claim valuation model, and discusses the alternative implementations of the numerical solutions. Section 3 examines how path dependency can be alleviated through state augmentation, allowing for path-dependent claims to be priced in the standard valuation framework. It also emphasizes the difficulty of solving the resulting degenerate equations with standard finite difference methods. Sections 4 and 5 establish necessary and sufficient conditions of holonomy for degenerate diffusions. Section 6 introduces the Forward Shooting Grid method which efficiently copes with this degeneracy, as well as with the early exercise condition of American path-dependent securities. Section 7 establishes the convergence of the FSG method for general multidimensional diffusion processes. Finally, Section 8 gives several examples of European and American prices for various path-dependent options, and discusses the results.

2 Contingent claim valuation

This section first briefly reviews the basic ingredients of the modern contingent claim valuation model in a continuous time framework (see e.g. Duffie (1992)), then discusses the alternative implementations of the numerical solutions.

2.1 The PDE approach

There exists an asset-price stochastic process S_t governing the evolution of the state variable S, which follows the Itô's stochastic differential equation (SDE):

$$dS_t \;=\; m(t,S_t)dt + b(t,S_t)dw,\; S_0 > 0, \tag{2.1}$$

where w is a standard Brownian motion.

There also exists a bond-price process B_t, governed by the SDE:

$$dB_t \;=\; r(t,S_t)B_t,\; B_0 > 0.$$

This process models the continuously compounding risk-free interest rate.

Consider a derivative security with terminal payoff $g(S_T)$, where g is some continuous real function that depends on state variable S. It is shown that in the absence of arbitrage, the price $C(t,S_t)$ of the derivative security at time t solves the partial differential equation (PDE):

$$-\,r(t,S)\,C(t,S) + \frac{\partial C}{\partial t}(t,S) + r(t,S)\,S\,\frac{\partial C}{\partial S}(t,S) + \frac{1}{2}b(t,S)^2\,\frac{\partial^2 C}{\partial S^2}(t,S) = 0 \tag{2.2}$$

with the boundary condition $C(T,S) = g(S)$.

2.2 The Feynman-Kac formula

Under suitable technical conditions for functions r, σ, and g, the Feynman-Kac (FK) formula gives the solution to eq. (2.2):

$$C(t,S) \;=\; E_t\left[\exp\left(-\int_t^T r(u,\hat{S}_u)du\right) g(\hat{S}_T)\right]; \tag{2.3}$$

where $\hat{S}_u$ is the Itô process defined by:

$$\begin{aligned} \hat{S}_u &= S_t,\; u \le t \\ d\hat{S}_u &= r(u,\hat{S}_u)\hat{S}_u du + b(u,\hat{S}_u)dw,\; u \ge t. \end{aligned}$$

Therefore, the value (price) $C(t,S)$ of a derivative security can be interpreted as the expectation of its discounted payoff under the modified (aka risk-neutral) process $\hat{S}_u$, whose expected rate of return is the riskless interest rate of the market.

Both the fundamental PDE (2.2) and its corresponding FK formula (2.3) extend to the case of a derivative security contingent to an arbitrary number of assets.

2.3 The Black-Scholes equation

In the Black-Scholes model, the stock price S follows a log-normal process, with r and σ constant:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dw, \ S_0 > 0 \\ dB_t &= r B_t, \ B_0 > 0. \end{aligned}$$

In this case, eq. (2.2) reduces to the Black-Scholes equation:

$$-r\, C(t,S) + \frac{\partial C}{\partial t}(t,S) + r\, S\, \frac{\partial C}{\partial S}(t,S) + \frac{1}{2}\sigma^2\, S^2\, \frac{\partial^2 C}{\partial S^2}(t,S) = 0, \tag{2.4}$$

with boundary condition $C(T,S) = g(S) = \max(0, S-K)$.

The change of variable

$$Z = \log(S/S_0) - \alpha t, \ \alpha = r - \sigma^2/2, \tag{2.5}$$

simplifies eq. (2.4) into:

$$-rC + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial Z^2} = 0, \tag{2.6}$$

with the boundary condition $C(T,Z) = \max(0, S_0 e^{Z+\alpha(T-t)} - K)$.

Looking for a solution of the form $C(t,Z) = e^{-r(T-t)} E(t,Z)$, eq. (2.6) reduces to the (backward) heat equation:

$$-\frac{\partial E}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 E}{\partial Z^2} = 0, \tag{2.7}$$

with the boundary condition $E(T,Z) = e^{r(T-t)} \max(0, S_0 e^{Z+\alpha(T-t)} - K)$.

Eq. (2.7) admits for solution the Black-Scholes option pricing formula:

$$\begin{aligned} C(t,S) &= S\Phi(X) - e^{-r(T-t)} K \Phi(X - \sigma\sqrt{T-t}) \\ X &= \frac{1}{\sigma\sqrt{T-t}} \left(\log(S/K) + (\alpha + \sigma^2)(T-t)\right), \end{aligned}$$

where Φ is the cumulative standard normal distribution function.

2.4 Numerical solutions

The PDE approach and FK formula give two related ways of pricing a derivative security when no closed form solution exists: either solve PDE (2.2) numerically, or use Monte Carlo simulation (Hammersley and Hanscomb (1964)) and sample the risk-neutral process given by eq. (2.1) in order to compute an approximation to the expectation of eq. (2.3). The respective advantages and limitations of these two methods are briefly presented below.

Monte Carlo simulation.

- The space complexity (memory requirement) is linear in the number of state variables. In general, the time complexity (computation time) is quadratic in the number of state variables.
- Due to the high number of samples usually required to get sufficient precision, execution times are significantly greater than those of finite difference methods.
- The early exercise condition of American options cannot be dealt with.

PDE integration. In the context of asset pricing problems, PDEs usually reduce to simple advection-diffusion equations, whose solutions are most simply computed through finite difference methods.

- Both the time and space complexities of FD methods grow exponentially in the number of state variables. This makes the PDE approach attractive only when the number of underlying assets is small.
- Numerical instabilities of FD methods can be a delicate issue. In particular, it is well-known that FD methods are ill suited to solving degenerate PDEs, that is PDEs for which the covariance matrix is singular.
- In general, FD implementations run significantly faster than corresponding Monte Carlo simulations.
- PDE solving methods are the only ones that easily handle the early exercise condition of American options[1].

3 Path dependency

This section first examines popular examples of path dependent problems, then describes how path dependency can be alleviated through state augmentation. It also emphasizes the difficulty of solving the resulting PDEs with standard FD methods.

3.1 A menagerie of path-dependent options

Historically, many similar path-dependent options have been given different names. This section presents the most popular path-dependent options, giving explicit formulas for their associated payoffs.

Without loss of generality, we will assume all options to be issued at initial time $t = 0$, expiring at time $T > 0$. We will write S_T the value of asset S at expiration time T, and K the strike or exercise price of the option. C_T (resp. P_T) will denote the payoff of a call (resp. put) at expiration time, and C (resp. P) the sought value of the call (resp. put) at initial time.

[1]The existence of solutions to advection diffusion PDE with early exercise boundary conditions has been studied by several authors McKean (1965); Merton (1973); Harrison and Kreps (1979); Bensoussan (1984); Karatzas (1988); Jaillet et al. (1988).

Let M_t (resp. m_t) be the maximum (resp. minimum) value of asset S over the time period $[0,t]$:

$$\begin{aligned} M_t &= \max_{0\leq\tau\leq t} S_\tau \\ m_t &= \min_{0\leq\tau\leq t} S_\tau . \end{aligned} \tag{3.1}$$

Let also A_t be the value of the arithmetic average of S over $[0,t]$:

$$A_t = \frac{1}{t}\int_0^t S_\tau d\tau \tag{3.2}$$

Ordinary option. An ordinary call (resp. put) option gives its owner the right to buy (resp. sell) S at strike K:

$$\begin{aligned} C_T &= \max(0, S_T - K) \\ P_T &= \max(0, K - S_T). \end{aligned}$$

Lookback. A lookback call (resp. put) gives its owner the right to buy (resp. sell) S at its lowest (resp. greatest) price over time period $[0,T]$:

$$\begin{aligned} C_T &= S_T - m_T \\ P_T &= M_T - S_T. \end{aligned}$$

Option on extrema. A call (resp. put) on maximum (resp. minimum) is like an ordinary option, but the spot is to be replaced with the historical maximum (resp. minimum) value of S:

$$\begin{aligned} C_T &= \max(0, M_T - K) \\ P_T &= \min(0, K - m_T). \end{aligned}$$

Capped options. A capped call (resp. put) option is like an ordinary option, as long as the historical maximum (resp. minimum) of S stays below (resp. above) a predefined upper (resp. lower) barrier price b. Should the maximum (resp. minimum) reach (resp. fall below) that barrier, the option is automatically exercised.

$$C_T = \begin{cases} b - K & \text{if } M_T \leq b \\ \max(0, S_T - K) & \text{otherwise} \end{cases}$$

$$P_T = \begin{cases} K - b & \text{if } m_T \geq b \\ \max(0, K - S_T) & \text{otherwise.} \end{cases}$$

Barrier options. Barrier options are also known as knock-out, knock-in, or trigger options. We describe below the most popular types.

- Down-and-out call. Up-and-out put.

  A down-and-out call (resp. up-and-out put) behaves like an ordinary call (resp. put) as long as the historical minimum (resp. maximum) of S stays above (resp. below) a predefined lower (resp. upper) barrier price b. Should the minimum (resp. maximum) reach or fall below (resp. reach or rise above) that barrier, the payoff becomes zero:

$$C_T = \begin{cases} \max(0, S_T - K) & \text{if } m_T > b \\ 0 & \text{otherwise} \end{cases}$$

$$P_T = \begin{cases} \max(0, K - S_T) & \text{if } M_T < b \\ 0 & \text{otherwise.} \end{cases}$$

- Down-and-in call. Up-and-in put.

  A down-and-in call (resp. up-and-in put) behaves like an ordinary call (resp. put) as long as the historical maximum (resp. minimum) of S stays below (resp. above) a predefined barrier upper price b. Should the maximum (resp. minimum) reach or rise above (resp. reach or fall below) that barrier, the payoff becomes zero:

$$C_T = \begin{cases} \max(0, S_T - K) & \text{if } M_T < b \\ 0 & \text{otherwise} \end{cases}$$

$$P_T = \begin{cases} \max(0, K - S_T) & \text{if } m_T > b \\ 0 & \text{otherwise} \end{cases}$$

Average-rate options. Average-rate options are also known as Asian options. Their payoffs depend on the value of the arithmetic average of S over a given time period. There are two types of such options: fixed-strike, and floating-strike.

- A fixed-strike average-rate option is like an ordinary option, with the time average A_T substituted for S_T:

$$\begin{aligned} C_T &= \max(0, A_T - K) \\ P_T &= \max(0, K - A_T). \end{aligned}$$

- Symmetrically, in the payoff of a floating-strike average-rate option, the time average A_T is substituted for the strike K:

$$\begin{aligned} C_T &= \max(0, S_T - A_T) \\ P_T &= \max(0, A_T - S_T). \end{aligned}$$

3.2 An introductory example

It should be noted that the PDE (2.2) for the price process can only be written down under the Markovian assumption that the instantaneous payoff of the derivative security be function only of the current value of asset S, i.e. $C = C(S_t, t)$. If this is not the case, the security is termed path-dependent. To understand the implications of such path-dependency with respect to the PDE solving approach, the following sections draw on an example developed in Stanton (1989) for the valuation of a simplified *zero-strike Asian option* in the Black-Scholes model.

Consider a stock S with no dividends, which follows a log-normal price process. Also, assume a constant risk-free interest rate. An option on S is issued at time $t = 0$, expiring at time $T > 0$, whose terminal payoff g_T is the arithmetic average A_T of S over period $[0, T]$, cf. eq. (3.2).

Applying the Feynman-Kac formula, the arbitrage-free price of this option can be computed as the discounted expectation of its payoff under the risk-neutral process $\hat{S}_u$:

$$\begin{aligned}\hat{S}_u &= S_t,\ u \leq t \\ d\hat{S}_u &= r\hat{S}_u du + \sigma \hat{S}_u dw,\ u \geq t;\end{aligned}$$

and

$$\begin{aligned}C(t) &= \frac{1}{T}e^{-r(T-t)}E_t\left[\int_0^T \hat{S}_u du\right] \\ &= \frac{1}{T}e^{-r(T-t)}\int_0^T E_t\left[\hat{S}_u\right] du.\end{aligned}$$

By definition of $\hat{S}_u$, $E_t\left[\hat{S}_u\right] = S_t$ for $u \leq t$. For $u \geq t$, $\hat{S}_u$ is a log-normal process, whose expectation at time t is given by:

$$E_t\left[\hat{S}_u\right] = S_t e^{r(u-t)}.$$

Therefore,

$$\begin{aligned}\int_0^T E_t\left[\hat{S}_u\right] du &= \int_0^t E_t\left[\hat{S}_u\right] du + \int_t^T E_t\left[\hat{S}_u\right] du \\ &= \int_0^t S_u du + \frac{S_t}{r}\left(e^{r(T-t)} - 1\right).\end{aligned}$$

Finally,

$$\begin{aligned}C(t) &= C(t, S_t, A_t) \\ &= \frac{e^{-r(T-t)}}{T}\int_0^t S_u du + \frac{S_t}{rT}\left(1 - e^{-r(T-t)}\right),\end{aligned} \tag{3.3}$$

which clearly shows that the value of the security, as well as its payoff are path-dependent.

3.3 Augmenting the state space

State augmentation is a classical method for converting path-dependent problems into their equivalent path-independent counterparts. The following example is drawn from Stanton (1989).

Let us therefore incorporate A as the second state variable. A_t being the average value of S over period $[0, t]$, $t \leq T$, with $A_0 = S_0$, the law of evolution of A_t is obtained by differentiating eq. (3.2):

$$dA_t = \frac{1}{t}(S_t - A_t)dt. \tag{3.4}$$

The value of the option at time t becomes $C = C(S, A, t)$. Using the same arbitrage arguments as in the regular case, it is shown that in absence of arbitrage, A_t solves the PDE:

$$-rC + \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{t}(S - A)\frac{\partial C}{\partial A} = 0, \tag{3.5}$$

with the boundary condition $C(T, S, A) = g = A$.

Eq. (3.5) is in fact the Black-Scholes equation (2.4), into which the term $1/t(S_t - A_t)\frac{\partial C}{\partial A}$ has been incorporated. By augmenting the state space, the value of the option only depends on the *current* values of the state variables S and A, hence path dependency has been alleviated. Furthermore, for this particular boundary condition, a closed-form formula for $C(t, S_t, A_t)$ is easily found:

$$C(t, S_t, A_t) \quad = \quad \frac{e^{-r(T-t)}}{T} tA_t + \frac{S_t}{rT}\left(1 - e^{-r(T-t)}\right), \tag{3.6}$$

which is exactly eq. (3.3) above.

In the case of a fixed-strike Asian option with a more realistic terminal payoff $g = \max(0, A - K)$, eq. (3.5) is still valid but the new boundary condition makes it impossible to find a closed-form formula for the price. In general, only numerical methods can handle path-dependent asset pricing problems, with the usual alternative of Monte Carlo simulation vs. PDE solving.

3.4 Degeneracy of augmented PDE

Augmented PDE are degenerate in nature. Intuitively, the degeneracy comes from the fact that the augmented state variables, such as A_t and S_t, are correlated. In other words, the increments dS_t and dA_t of asset price and time average are not independent, which implies that the deterministic relationship between them, as expressed in eq. (3.5) cannot be readily integrated. In practice, any numerical scheme that approximates an augmented PDE, although it can be made stable, will not necessarily converge to a solution. This makes the standard FD approach impractical. Again, we shall illustrate this with the example of the zero-strike Asian option, eq. (3.6) providing analytic solutions for reference.

In eq. (3.5), a discontinuity occurs at $t = 0$, which can be eliminated by the change of variable $Y = tA$. Together with the change indicated in eq. (2.5), they simplify eq. (3.5) into:

$$\frac{\partial E}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 E}{\partial Z^2} + S\frac{\partial E}{\partial Y} = 0, \tag{3.7}$$

with the boundary condition $E(T, Z, Y) = e^{-rT} Y/T$.

Because there is no second-order term in Y, eq. (3.7) is a strongly degenerate two-dimensional advection-diffusion PDE. invert a singular matrix. Writing $E(t, Z, Y)$ as $E^n_{j,k} = E(t_n, Z^n_j, Y^n_k)$, eq. (3.7) is approximated with the following explicit scheme:

$$-D_t(t) \quad = \quad \frac{1}{2}\sigma^2 D_{11}(t+\Delta t) + S(t+\Delta t) D_2(t+\Delta t),$$

where

$$\begin{aligned}
D_t(t) &= \frac{E^n_{j,k} - E^{n+1}_{j,k}}{\Delta t} &&= -\frac{\partial}{\partial t}E + O(\Delta t) \\
D_2(t) &= \frac{E^n_{j,k+1} - E^n_{j,k-1}}{2\Delta Y} &&= \frac{\partial}{\partial Y}E + O(\Delta Y^2) \\
D_{11}(t) &= \frac{E^n_{j+1,k} - 2E^n_{j,k} + E^n_{j-1,k}}{\Delta Z^2} &&= \frac{\partial^2}{\partial Z^2}E + O(\Delta Z^2).
\end{aligned}$$

Eq. (3.7) is then backward integrated as follows:

$$\begin{aligned}
u_Z &= \frac{\sigma^2 \Delta t}{\Delta Z^2} \\
u_Y &= S^{n+1}_j \frac{\Delta t}{\Delta Y}; \\
E^n_{j,k} &= (1-u_Z)E^{n+1}_{j,k} + \tfrac{1}{2}u_Z(E^{n+1}_{j+1,k} + E^{n+1}_{j-1,k}) + \tfrac{1}{2}u_Y(E^{n+1}_{j,k+1} - E^{n+1}_{j,k-1}) \\
n &= N-1, \ldots, 0 \\
j &= -n, \ldots, n \\
k &= 0, \ldots, k_m
\end{aligned} \tag{3.8}$$

with the boundary condition

$$\begin{aligned}
E^N_{j,k} &= e^{-rT} Y^N_k / T \\
j &= -N, \ldots, N \\
k &= 0, \ldots, k_m.
\end{aligned}$$

We implemented the scheme (3.8), and compared results with those obtained with formula (3.6). Not surprisingly, the numerical procedure failed to give accurate results on a broad range of input parameters σ and r.

In summary, although path dependency can be alleviated by augmenting the state space, the applicability of FD methods in such cases is very limited. This seems to give a definite advantage to Monte Carlo methods, which just simulate the asset-price process. This fact has been noted in previous work (Talay (1991)). In Section 6, we will present an alternative numerical technique for solving degenerate PDEs that has several advantages over the Monte-Carlo method. Before, we derive a mathematical characterization of path-dependence.

4 Characterization of path-dependent price processes in two dimensions

We establish necessary and sufficient conditions under which degenerate diffusion PDE can be reduced to lower-dimensional non-degenerate PDE on a sub-manifold of the state space. A degenerate PDE that can be reduced in such a way is called *holonomic*. It is called *non-holonomic* otherwise. It is important to determine whether a given PDE is holonomic or not, since this property influences both the type of numerical integration technique that can be used, and the memory required to conduct the integration. We introduce in this section the main intuition behind this characterization in the two dimensional case. In the next section, we treat the general case.

4.1 The generic two-dimensional path-dependent pricing problem

We consider a price process S_t following the Itô SDE (2.1), and an arbitrary path-dependent variable A_t:

$$A_t = \Psi\left([S_\tau]_{\tau \leq t}\right)$$

We assume that A_t follows an Itô SDE of the type:

$$dA_t = m_A(t, S_t, A_t)dt + b_A(t, S_t, A_t)dw$$

For example, if A_t is the mean value of S_t up to time t, we see from the previous section that:

$$m_A(t, S, A) = \frac{1}{t}(S - A), \quad b_A(t, S, A) = 0$$

In order to compute the price of a path-dependent contingent claim $C(t, S_t, A_t)$, we must solve for the following two-dimensional degenerate PDE:

$$-rC + \frac{\partial C}{\partial t} + m_S\frac{\partial C}{\partial S} + \frac{1}{2}b_S^2\frac{\partial^2 C}{\partial S^2} + m_A\frac{\partial C}{\partial A} + \frac{1}{2}b_A^2\frac{\partial^2 C}{\partial A^2} + b_S b_A\frac{\partial^2 C}{\partial S\partial A} = 0 \tag{4.1}$$

where: $m_S = rS, \quad b_S = b.$

The numerical integration of the above equation requires *a priori* to quantize both variables S and A. Hence, if each variable is quantized using N samples, the memory requirement of the numerical pricing procedure is proportional to N^2.

However, since the time evolutions of S and A depend on the same Brownian motion W, there is a simple relationship between the three (stochastic) differentials dt, dS, and dA:

$$d\omega = (m_S b_A - b_S m_A)dt - b_A dS + b_S dA = 0 \tag{4.2}$$

The numerical integration of the PDE will require a memory proportional to N^2 only if this differential relationship cannot be integrated, i.e. if there is no *potential function* $F(t, S, A)$ and no *integrating factor* $\lambda(t, S, A)$ such that:

$$dF = \lambda d\omega$$

Indeed, if such F and λ exist, then the differential relationship (4.2) is equivalent to the following:

$$F(t, S_t, A_t) = F(0, S_0, A_0)$$

Hence, we can perform (under suitable technical conditions) the following change of variables in PDE (4.1):

$$\begin{array}{rcccl} \tau & = & \Phi_t(t, S, A) & = & t \\ \Sigma & = & \Phi_S(t, S, A) & = & S \\ F & = & \Phi_F(t, S, A) & = & F(t, S, A) \end{array}$$

and get:

$$-rC + \frac{\partial C}{\partial \tau} + m_S \frac{\partial C}{\partial \Sigma} + \frac{1}{2} b_S^2 \frac{\partial^2 C}{\partial \Sigma^2} + m_F \frac{\partial C}{\partial F} + \frac{1}{2} b_F^2 \frac{\partial^2 C}{\partial F^2} + b_S b_F \frac{\partial^2 C}{\partial \Sigma \partial F} = 0 \tag{4.3}$$

where we have defined for notational convenience:

$$m_F = \frac{\partial F}{\partial t} + m_S \frac{\partial F}{\partial S} + \frac{1}{2} b_S^2 \frac{\partial^2 F}{\partial S^2} + m_A \frac{\partial F}{\partial A} + \frac{1}{2} b_A^2 \frac{\partial^2 F}{\partial A^2} + b_S b_A \frac{\partial^2 F}{\partial S \partial A}$$

and

$$b_F = b_S \frac{\partial F}{\partial S} + b_A \frac{\partial F}{\partial A}$$

But by Itô's formula:

$$dF = m_F dt + b_F dw$$

Since by assumption $dF = \lambda d\omega = 0$, we get $m_F = b_F = 0$. Hence, the transformed PDE (4.3) simplifies to the usual one-dimensional Black-Scholes equation:

$$-rC + \frac{\partial C}{\partial \tau} + m_S \frac{\partial C}{\partial \Sigma} + \frac{1}{2} b_S^2 \frac{\partial^2 C}{\partial \Sigma^2} = 0 \tag{4.4}$$

In order to price a contingent claim with terminal payoff $g(S_T, A_T)$, one must first solve for $A = \phi(t, S)$ as a function of t and S in equation $F(t, S_t, A_t) = F(0, S_0, A_0)$, and then solve backwards in time the above one-dimensional PDE with the boundary condition $C(T, S, A) = g(S, \phi(T, S))$. The memory required to conduct the numerical integration is now proportional to N instead of N^2. Also, the above one dimensional PDE is not degenerate, and can be integrated using standard FD methods.

4.2 Fisk-Stratonovitch stochastic differentials

It is an important matter to characterize the conditions under which the differential relationship (4.2) can be integrated. For that purpose, we will first introduce the notion of stochastic differential in the *Fisk-Stratonovich* sense.

Let $0 = t_0 < t_1 < \ldots < t_m = t$ be an arbitrary partition of the interval $[0, t]$ with a modulus $\eta = \max_i(t_{i+1} - t_i)$.

Formally, the *Fisk-Stratonovitch* SDE $dx = b \circ dw$ for two continuous semi-martingales b, w can be interpreted as a notation for the identity

$$\forall t, \quad x(t) = x(0) + \lim_{\eta \longrightarrow 0} \sum_{j=0}^{m-1} \frac{b(t_j) + b(t_{j+1})}{2} (w(t_{j+1}) - w(t_j))$$

where the limit is taken in probability.

Intuitively, whereas the Itô differential $dx = bdw$ is a *forward differential* which can be loosely interpreted as:

$$x(t+dt) = x(t) + b(t)(w(t+dt) - w(t))$$

the Fisk-Stratonovitch differential $dx = b \circ dw$ is a *symmetric differential*, which can be loosely interpreted as:

$$x(t+dt) = x(t) + \frac{b(t) + b(t+dt)}{2}(w(t+dt) - w(t))$$

It can be shown (see e.g. Karatzas and Shreve (1988)) that, unlike Itô differentials which follow Itô's chain rule, Fisk-Stratonovitch differentials follows the chain rule of the classical differential calculus. In particular, for any smooth real-valued function $f(t, x_1, \ldots, x_d)$ and continuous vectors of semi-martingales $X = (x_1, \ldots, x_d)$, we have:

$$df(t, X) = \frac{\partial f}{\partial t}(t, X)dt + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(t, X) \circ dx_i$$

Finally, it can be shown that there is the following relationship between Itô and Fisk-Stratonovitch SDE. If $X = (x_1, \ldots, x_d)$ follows the Itô SDE:

$$\forall i \in [1, d], \quad dx_i = m_i(t, X)dt + \sum_{j=1}^{k} b_{ij}(t, X)dw_j$$

Then

$$\forall i \in [1, d], \quad dx_i = \left(m_i - \frac{1}{2} \sum_{l=1}^{d} \sum_{j=1}^{k} b_{lj} \frac{\partial b_{ij}}{\partial x_l} \right) dt + \sum_{j=1}^{k} b_{ij} \circ dw_j$$

4.3 The stochastic Frobenius integrability condition in two dimensions

In this subsection, we introduce the integrability conditions for equation (4.2). We consider for simplicity the case $d = 2$, $k = 1$. The system of Itô SDE:

$$\begin{aligned} dx_1 &= m_1(t, x_1, x_2)dt + b_1(t, x_1, x_2)dw \\ dx_2 &= m_2(t, x_1, x_2)dt + b_2(t, x_1, x_2)dw \end{aligned}$$

admits the following Fisk-Stratonovitch equivalent:

$$\begin{aligned} dx_1 &= \tilde{m}_1(t, x_1, x_2)dt + b_1(t, x_1, x_2) \circ dw \\ dx_2 &= \tilde{m}_2(t, x_1, x_2)dt + b_2(t, x_1, x_2) \circ dw \end{aligned}$$

with

$$\tilde{m}_1 = m_1 - \frac{1}{2}\left(b_1\frac{\partial b_1}{\partial x_1} + b_2\frac{\partial b_1}{\partial x_2}\right)$$
$$\tilde{m}_2 = m_2 - \frac{1}{2}\left(b_1\frac{\partial b_2}{\partial x_1} + b_2\frac{\partial b_2}{\partial x_2}\right)$$

By eliminating dw, we get the following differential relationship:

$$d\omega = \omega_0 dt + \omega_1 \circ dx_1 + \omega_2 \circ dx_2 = (\tilde{m}_1 b_2 - \tilde{m}_2 b_1)dt - b_2 \circ dx_1 + b_1 \circ dx_2 = 0$$

By the Fisk-Stratonovitch (i.e. classical) chain rule, a sufficient condition for the existence of a potential function $F(t, x_1, x_2)$ and an integrating factor $\lambda(t, x_1, x_2)$ such that $dF = \lambda d\omega$ is:

$$\frac{\partial F}{\partial t} = \lambda\omega_0, \quad \frac{\partial F}{\partial x_1} = \lambda\omega_1, \quad \frac{\partial F}{\partial x_2} = \lambda\omega_2 \tag{4.5}$$

From the above relations and the symmetry of the second order mixed partial derivatives of F, we get after elimination of λ:

$$\omega_0\left(\frac{\partial\omega_2}{\partial x_1} - \frac{\partial\omega_1}{\partial x_2}\right) + \omega_1\left(\frac{\partial\omega_2}{\partial t} - \frac{\partial\omega_0}{\partial x_2}\right) + \omega_2\left(\frac{\partial\omega_1}{\partial t} - \frac{\partial\omega_0}{\partial x_1}\right) = 0 \tag{4.6}$$

It can be shown (see section 5) that the above relation is a necessary and sufficient condition for the existence of F. We can express the above condition in terms of m_i and b_j instead of ω_i. Let us denote $B_0 = (1, \tilde{m}_1, \tilde{m}_2)^T$ and $B_1 = (0, b_1, b_2)^T$. After elementary algebraic manipulations, the integrability condition (4.6) is shown to be equivalent to:

$$\det\left(B_0, B_1, [B_0, B_1]\right) = 0$$

where $[B_0, B_1]$ is the *Lie Bracket* (see also section 5) of the vector fields B_0 and B_1, i.e. the vector field:

$$[B_0, B_1] = dB_0.B_1 - dB_1.B_0$$

and where dB_i denote the Jacobian matrix of the vector field B_i.

4.4 Characterization of path dependent prices processes

We consider the path-dependent price process of subsection (4.1). We take $x_1 = S$ and $x_2 = A$.

Define

$$\tilde{m}_S = m_S - \frac{1}{2}b_S\frac{\partial b_S}{\partial S}$$
$$\tilde{m}_A = m_A - \frac{1}{2}\left(b_S\frac{\partial b_A}{\partial S} + b_A\frac{\partial b_A}{\partial A}\right)$$

and let $B_0 = (1, \tilde{m}_S, \tilde{m}_A)^T$, and $B_1 = (0, b_S, b_A)^T$. Then:

> *The degenerate two-dimensional Black-Scholes PDE (4.1) in the variables (S,A) can be reduced to the one-dimensional non-degenerate PDE (4.4) if and only if the following condition is satisfied:*

$$\det\left(B_0, B_1, [B_0, B_1]\right) = 0$$

We can apply the above result to the case of a path-dependent variable A which is instantaneously riskless. i.e. $b_A = 0$.

We have: $B_0 = (1, \tilde{m}_S, \tilde{m}_A)^T$, $B_1 = (0, b_S, 0)^T$, hence

$$det\left(B_0, B_1, [B_0, B_1]\right) = -b_s^2 \frac{\partial m_A}{\partial S}$$

Therefore, the problem can be reduced to a one-dimensional problem iff m_A does not depend on S. This is obviously not the case for the average-rate option pricing problem:

$$\frac{\partial m_A}{\partial S} = \frac{1}{t} \neq 0$$

We can state:

> *The average-rate option pricing PDE (3.5) cannot be reduced to a one-dimensional non-degenerate PDE.*

5 Characterization of path-dependent price processes: general case

In this section, we turn to a generalization of the above result.

5.1 The general diffusion problem

We consider an advection-diffusion equation of the type:

$$-\frac{\partial f}{\partial t} = -rf + \sum_{i=1}^{d} m_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} \gamma_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \tag{5.1}$$

where $X = (x_1, \ldots, x_d)^T \in R^d$, $M = (m_1, \ldots, m_d)^T$, with the following boundary condition:

$$f(T, x_1, \ldots, x_d) = g(x_1, \ldots, x_d)$$

We assume that the real valued functions $r(t, x_1, \ldots, x_d) > 0$, $m_i(t, x_1, \ldots, x_d)$, $i \in [1, d]$, and $\gamma_{ij}(t, x_1, \ldots, x_d)$, $(i, j) \in [1, d]^2$ on $[0, \infty[\times R^d$ are bounded and sufficiently smooth. We assume furthermore that for any $t, x_1, \ldots, x_d$ the matrix:

$$\Gamma(t, x_1, \ldots, x_d) = \left(\gamma_{ij}(t, x_1, \ldots, x_d)\right)_{(i,j)\in[1,d]^2}$$

is symmetric and non-negative. We denote by B its Cholesky decomposition:

$$\Gamma = BB^T$$

B is a $d \times k$ matrix of rank k, with $k \leq d$.

By the Feynman-Kac formula, the solution of the above PDE can be expressed:

$$f(t, x_1, \ldots, x_d) = E_t\left(\exp\left(-\int_0^T r(\tau, X(\tau))d\tau\right) g(X(T))\right)$$

where X is the solution of the SDE:

$$dX = Mdt + BdW, \quad X(t) = (x_1, \ldots, x_d)^T \tag{5.2}$$

$W = (w_1, \ldots, w_k)$ is a k-dimensional standard Brownian motion.

We can define the vector $\tilde{M}$:

$$\forall i \in [1, d], \quad \tilde{m}_i = m_i - \frac{1}{2}\sum_{l=1}^{d}\sum_{j=1}^{k} b_{lj}\frac{\partial b_{ij}}{\partial x_l}$$

Then, from subsection (4.2):

$$dX = \tilde{M}dt + B \circ dW \tag{5.3}$$

We denote by $S^W(X_0)$ the *support* of the above SDE for $X(0) = X_0$, i.e. the smallest closed set of paths $X(t)$ (elements of the Wiener space Karatzas and Shreve (1988); Ikeda and Watanabe (1981)) such that $\text{Prob}(S^W(X_0)) = 1$. The *set of accessible states at time t* of SDE (5.3) for $X(0) = X_0$, i.e. the set of values $X(t)$ for elements of $S^W(X_0)$, is denoted by $A^W(t, X_0)$. In other words, $A^W(t, X_0)$ is the *time t projection* of $S^W(X_0)$. Clearly,

$$\text{Prob}(X(t) \in A^W(t, X_0)) = 1$$

The *set of accessible states from* X_0 for system (5.3) is defined as:

$$A^W(X_0) = \cup_{t \geq 0} A^W(t, X_0)$$

Clearly,

$$\text{Prob}(\forall t, X(t) \in A^W(X_0)) = 1$$

5.2 The certainty equivalence theorem of Stroock and Varadhan

The following theorem was first established by Stroock and Varadhan (1972). We use here the formulation of Ikeda and Watanabe, where the theorem is established as a consequence of general convergence results for approximations of diffusion processes (see Ikeda and Watanabe (1981)).

We consider the *deterministic* control system:

$$\frac{dX}{dt} = \tilde{M} + B\frac{dU}{dt} \tag{5.4}$$

where $U(t) = (u_1(t), \ldots, u_k(t))$ is a piecewise smooth function of time.

We define the support $S^U(X_0)$ of system (5.4) as the set of all possible paths $X(t)$ starting at X_0 for all possible control paths U. Similarly, we define the set $A^U(t, X_0) \subset R^d$ of *accessible states at time t* from X_0 as the set of states $X^* \in R^d$ such that there exist a control $U(t)$ verifying $X^* = X(t)$. The set $A^U(X_0) \subset R^d$ of accessible states from X_0 is similarly defined as $A^U(X_0) = \cup_{t \geq 0} A^U(t, X_0)$. Finally, we say that a state X^* is *weakly accessible* from X_0 if there exist a sequence $X_0, X_1, \ldots, X_n = X^*$ such that either X_{i+1} is accessible from X_i, or X_i is accessible from X_{i+1}. The set of states weakly accessible from X_0 is denoted $WA(X_0)$. By definition, any accessible state is weakly accessible, i.e. $A(X_0) \subset WA(X_0)$. For a non-symmetric system, i.e. a system such that the controls U cannot always be inverted, then there may be weakly accessible states that are not accessible.

We have the following fundamental result (Stroock and Varadhan (1972)):

(Stroock and Varadhan, 1972)

For any X_0, the support of the Fisk-Stratonovitch system (5.3) is the closure of support of the deterministic system (5.4):

$$S^W(X_0) = \overline{S^U(X_0)}$$

We will only use the following immediate corollary:

For any X_0, and any time t, the set of accessible states at time t from X_0 of the Fisk-Stratonovitch system (5.3) is the closure of the set of accessible states at time t from X_0 of the deterministic system (5.4):

$$\forall t \geq 0, \quad \forall X_0, \quad A^W(t, X_0) = \overline{A^U(t, X_0)}$$

5.3 Frobenius integrability condition and Chow's theory

In the previous section, we used the certainty equivalence theorem in order to establish a link between stochastic and deterministic accessibility for differential equations. In this section, we apply standard result on the accessibility of deterministic control systems to characterize the integrability of degenerate SDE.

We consider again the deterministic system (5.4). For the purpose of the following discussion, we will include the time variable t in the state, i.e. we consider the state of the system at time t to be $Y = (y_0(t), y_1(t), \ldots, y_d(t)) = (t, x_1(t), \ldots, x_d(t)) \in R^{d+1}$. If b_{ij} denote the elements of the matrix B, we define the extended $(d+1) \times (d+1)$ matrix $\tilde{B}$:

$$\tilde{B} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ \tilde{m}_1 & b_{11} & \ldots & b_{1d} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ \tilde{m}_d & b_{d1} & \ldots & b_{dd} \end{pmatrix}$$

Then, system (5.4) is equivalent to the following system in R^{d+1}:

$$\frac{dY}{dt} = \tilde{B}\frac{d\tilde{U}}{dt} \tag{5.5}$$

where $\tilde{U}(t) = (t, U(t)) = (t, u_1(t), \ldots, u_k(t)) = (\tilde{u}_0(t), \tilde{u}_1(t), \ldots, \tilde{u}_k(t))$ is the extended control.

Let $A^{\tilde{U}}(Y_0)$ denote the set of extended states accessible from $Y_0 = (0, X_0)$. Clearly:

$$A^{\tilde{U}}(Y_0) = \cup_{t\geq 0}\{t\} \times A^U(t, X_0)$$

and

$$WA^{\tilde{U}}(Y_0) = \cup_{t\geq 0}\{t\} \times WA^U(t, X_0)$$

By definition, any accessible state is weakly accessible. Reciprocally, if the drift is zero, i.e. $\tilde{M} = (\tilde{m}_1, \ldots, \tilde{m}_d) = 0$, then weak accessibility is equivalent to accessibility. Indeed, since the system (5.5) is symmetric in the control variable U (i.e. $-U$ is an admissible control iff U is an admissible control), the extended control $\tilde{U}$ becomes symmetric for a zero drift.

If $\tilde{M} = 0$, we get:

$$\forall t, \quad WA^U(t, Y_0) = A^U(t, X_0)$$

We now recall the definition of the Lie bracket of two vector fields. Let (V^1, V^2) be any pair of vector fields in R^{d+1}.

Given any point $Y_0 = (t, X_0) \in R^{d+1}$, let us consider a path starting at Y_0 and obtained by concatenating the four following paths:

- the first path follows the flow[2] of V^1 during δt;

- the second path follows the flow of V^2 during δt;

- the third path follows the flow of $-V^1$ during δt;

- the fourth path follows the flow of $-V^2$ during δt.

Let $Y_1 = (t + \delta t, X_1)$ be the configuration reached at the end of these four paths. A straightforward Taylor expansion shows that:

$$\lim_{\delta t \to 0} \frac{Y_1 - Y_0}{\delta t^2} = dV^2 \cdot V^1 - dV^1 \cdot V^2,$$

where $dV^2 \cdot V^1$ and $dV^1 \cdot V^2$ denote the products of the $(d+1) \times (d+1)$ Jacobian matrices:

[2]The *integral curve* of a vector field V is a curve whose tangent at every point Y is $V(Y)$. We say that the curve *follows the flow* of V.

$$dV^2 = \begin{pmatrix} \frac{\partial v_0^2}{\partial y_0} & \dots & \frac{\partial v_0^2}{\partial y_d} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial v_d^2}{\partial y_0} & \dots & \frac{\partial v_d^2}{\partial y_d} \end{pmatrix}; \quad dV^1 = \begin{pmatrix} \frac{\partial v_0^1}{\partial y_0} & \dots & \frac{\partial v_0^1}{\partial y_d} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial v_d^1}{\partial y_0} & \dots & \frac{\partial v_d^1}{\partial y_d} \end{pmatrix},$$

and the $(d+1)$-vectors:

$$V^1 = (v_0^1, v_1^1 \quad \dots \quad v_d^1)^T; \qquad V^2 = (v_0^2 \quad v_1^2 \quad \dots \quad v_d^2)^T.$$

The expression $dV^2 \cdot V^1 - dV^1 \cdot V^2$ determines a new vector field which is commonly denoted by $[V^1, V^2]$ and called the *Lie bracket* of V^1 and V^2.

Let $B_0, \dots, B_d$ be the columns of the extended matrix $\tilde{B}$. By definition, the Control Lie Algebra associated with system (5.5), denoted by $CLA(\tilde{B})$, is the smallest Lie algebra which contains $B_0, \dots, B_d$. Stated otherwise, $CLA(\tilde{B})$ is the subspace of vector fields generated by all the linear combinations of vector fields $B_0, \dots, B_d$ and all their Lie brackets recursively computed.

For every $Y_0 \in R^{d+1}$, let $CLA(\tilde{B})(Y_0)$ denote the subspace of vectors spanned by the vector fields of $CLA(\tilde{B})$ at Y_0. A connected sub-manifold $\mathcal{M}$ of R^{d+1} is an *integral sub-manifold* of $CLA(\tilde{B})$ if at each $Y \in R^{d+1}$ the tangent space to $\mathcal{M}$ is contained in $CLA(\tilde{B})(Y)$. $\mathcal{M}$ is a *maximal integral sub-manifold* of $CLA(\tilde{B})$ if it is not properly included in any other integral manifold.

The Frobenius integrability theorem can be stated as follows:

Frobenius Integrability Theorem *If the dimension of $CLA(\tilde{B})(Y)$ has a constant value r for every $Y \in R^{d+1}$, there exists a partition of R^{d+1} into maximal integral sub-manifolds of $CLA(\tilde{B})$ all of dimension r.*

The maximal integral sub-manifold of dimension r passing through Y is denoted $\mathcal{M}_r(Y)$.

The following results derive from the work of Chow (1939). They were elucidated in Hermann (1963); Haynes and Hermes (1970); Lobry (1970); Sussmann and Jurdjevic (1972); Hermann and Krener (1977). We follow the presentation of Hermann and Krener (1977).

Chow's Theorem *If the dimension of $CLA(\tilde{B})(Y)$ has a constant value r for every $Y \in R^{d+1}$, then*

$$\forall Y \in R^{d+1}, \ \ \mathcal{M}_r(Y) = WA^U(Y)$$

Furthermore, the interior of $A^U(Y)$ as a subset of $\mathcal{M}_r(Y) = WA^U(Y)$ is not empty.

5.4 A characterization of holonomy for degenerate diffusions

By inspection of the extended matrix $\tilde{B}$, and since the original matrix B is by definition regular, the minimal dimension of the Control Lie Algebra is $r_{min} = k+1$.

Let us assume that the Control Lie Algebra has a constant dimension $k+1 \leq r \leq d+1$. We will show that there exist exactly $m = d+1-r$ constraints satisfied almost surely by the solution of the Fisk-Stratonovitch system (5.3).

We will first show $m \geq d+1-r$. Indeed, Chow's Theorem states that the set of weakly accessible states for (5.5) is a sub-manifold of dimension r. This means that there exist a set of $d+1-r$ independent constraints $F = \big(F_1(t, x_1, \ldots, x_d), \ldots, F_{d+1-r}(t, x_1, \ldots, x_d)\big)$ such that

$$\forall t \in R,\ \ \forall X_0 \in R^d, \quad WA^U(t, X_0) = \{X \in R^d,\ \ F(t, X) = F(0, X_0)\}$$

This result, together with the Certainty Equivalence Theorem, implies that the set of accessible states from X_0 of the Fisk-Stratonovitch system (5.3) satisfies the constraints $F(t, X) = F(0, X_0)$ almost surely. Indeed, the set of accessible states is a subset of the set of weakly accessible states. Hence $m \geq d+1-r$.

Reciprocally, let us assume that the solution $X(t)$ of the Fisk-Stratonovitch system (5.3) satisfies almost surely m independent constraints $F = (F_1, \ldots, F_m)$. By the certainty equivalence theorem, we have $F(A^{\tilde{U}}(Y_0)) = F(Y_0)$. But, by Chow's theorem, the interior of $A^{\tilde{U}}(Y_0)$ in $\mathcal{M}_r$ is not empty. Hence, it cannot satisfy more than $d+1-r$ constraints, since $\mathcal{M}_r$ is of dimension r. This implies $m \leq d+1-r$.

We can state:

> **Condition for the existence of $d+1-r$ constraints**
>
> *If the Control Lie Algebra $CLA(\tilde{B})$ has constant dimension r, then there exist* exactly *$d+1-r$ independent constraints $F = (F_1, \ldots, F_{d+1-r})$ such that the solution of the Fisk Stratonovitch system (5.3) satisfies the constraints almost surely.*
>
> $$Prob\,\big(\forall t, F(t, X(t)) = F(0, X_0)\big) = 1$$

A degenerate Itô SDE of the form (5.2) and a degenerate advection-diffusion PDE on the form (5.1) are called *holonomic* if they can be reduced to lower dimensional non-degenerate counterparts on an appropriate sub-manifold of R^d. They are called non-holonomic otherwise. We have the following characterization of holonomy:

> **Characterization of holonomy for degenerate diffusions**
>
> *Consider a degenerate d-dimensional Itô SDE of rank $k < d$ of the form (5.2) and its corresponding degenerate advection-diffusion PDE on the form (5.1). Assume that the control Lie algebra $CLA(\tilde{B})$ has a constant dimension r.*
>
> *Then, the Itô SDE (5.2) and the corresponding PDE (5.1) are holonomic if and only if $r = k+1$*

When dealing with a degenerate PDE of the form (5.1), we must first compute the dimension of the control Lie algebra. If the dimension is minimal $(k+1)$, then we must change variables, replacing the last $d-k$ variables by the $d-k$ integrable constraints $F_1, \ldots, F_{d-k}$. Then, the PDE is transformed into a non-degenerate PDE on the k dimensional integral sub-manifold satisfying the constraints. Hence, standard FD methods can be used for the numerical integration.

On the other hand, if the dimension of the control Lie algebra is not minimal $(r > k+1)$, then the PDE is *non-holonomic*, i.e. intrinsically degenerate. Then, FD methods are not appropriate. In the next section, we describe an alternative numerical integration technique for solving non-holonomic diffusion PDE. We first describe the method for typical two-dimensional path-dependent pricing problems such as the average-rate option pricing problem. Then, we present the method in full generality.

6 The Forward Shooting Grid (FSG) method

This section introduces a new numerical method for the path-dependent asset pricing problem, which we call *the Forward Shooting Grid* method (FSG). FSG efficiently copes with the degeneracy of augmented PDEs. More generally, the FSG method is adequate for solving any non-holonomic advection-diffusion equation.

The FSG method was first introduced (under a different name) by Barraquand and Latombe (1993) for solving non-holonomic Hamilton-Jacobi-Bellman equations in a deterministic setting. However, to the best of our knowledge, the FSG method has never been used before for solving non-holonomic stochastic optimal control problems such as the advection-diffusion problem described here.

6.1 Principle

The FSG method consists in taking advantage of the forward SDE equations that govern the correlated evolutions of the augmented state variables with respect to the underlying asset variables. Combined with an *a priori* quantization of the augmented state, those equations allow to construct the discrete state graph of the correlated variables, which is used in turn to integrate backwards in time the degenerated equation in the state variables.

Consider a path-dependent contingent claim on an asset S, with terminal payoff $C_T = g(S_T, A_T)$, where A is a path-dependent variable, e.g. the historical minimum or average of S. Also, assume S follows an Itô SDE, such as in eq. (2.1).

By augmenting the state space with A, the price C of the claim depends on both S and A. Let us define an *a priori* quantization as follows. Given a time step Δt, we fix two values ΔY, ΔZ, and two invertible quantization functions $\overline{S}$, $\overline{A}$, such that:

$$\begin{array}{rcl} S_t &=& \overline{S}(n\Delta t, j\Delta Z) \qquad = \quad S_j^n \\ A_t &=& \overline{A}(n\Delta t, k\Delta Y) \qquad = \quad A_k^n \\ n &=& 0, \ldots, N = T/\Delta t \\ j &=& j_0(n), \ldots j_m(n) \\ k &=& k_0(n), \ldots k_m(n). \end{array} \tag{6.1}$$

Second, let us assume a function ψ quantifying the correlated evolutions of A with respect to S, i.e. for an arbitrary time step Δt:

$$A_{t+\Delta t} \quad = \quad \psi(A_t, S_{t+\Delta t}). \tag{6.2}$$

Eq. (6.2) relates the variations of S and A upon a transition from state (S_t, A_t) to state $(S_{t+\Delta t}, A_{t+\Delta t})$. Under the quantization (6.1), it has the following discrete equivalent:

$$A_{k_{new}}^{n+1} \quad = \quad \psi\left(A_k^n, S_{j_{new}}^{n+1}\right) \tag{6.3}$$

where j_{new} (resp. k_{new}) denote the set of all j (resp. k) values used to approximate state $(S_{t+\Delta t}, A_{t+\Delta t})$ of the transition. By assumption, the law of evolution of S is known, hence j_{new} values can be obtained from the approximation of the corresponding SDE (e.g. through a Cox, Ross, Rubinstein (CRR) binomial approximation). In order to get the corresponding k_{new}, we invert the quantization function $\overline{A}$ at time $(n+1)\Delta t$, and take the resulting nearest integer value:

$$\begin{array}{rcl} k_{new} &=& \text{nearest}\left[\dfrac{\overline{A}^{-1}\left(\psi\left(A_k^n, S_{j_{new}}^{n+1}\right)\right)}{\Delta Y}\right] \\ &=& \phi(k, j_{new}). \end{array} \tag{6.4}$$

In other words, we obtain k_{new} by shooting the best approximating A bucket forward in time through function ϕ, hence the name of the method.

The last step consists in finding the law of evolution for $C_{j,k}^n = C(n\Delta t, S_j^n, A_k^n)$. By the Feynman-Kac formula, the price $C(0, S, A)$ can be computed as the discounted expectation of its payoff future under the risk neutral process $(\hat{S}_u, \hat{A}_u)$. Choosing a CRR binomial approximation, we have $j_{new} = \{j+1, j-1\}$. We let $k_{new} = \{k+, k-\}$, where $k- = \phi(k, j-1)$, and $k+ = \phi(k, j+1)$. This gives:

$$C_{j,k}^n \quad = \quad u\, C_{j+1,k+}^{n+1} + (1-u)\, C_{j-1,k-}^{n+1}, \tag{6.5}$$

where $u = 1/2 + \alpha\Delta t/2\sigma^2$ is the associated risk-neutral probability.

The pricing algorithm proceeds in two steps by first building the discrete state lattice induced by the binomial approximation, using the forward integration of eq. (6.3), then computing the price C, using the backward integration of eq. (6.5). It is straightforward to check for the early exercise condition at each time step n of this scheme.

6.2 Application

This section details the numerical valuation of two typical path-dependent instruments: a lookback call, and an average-rate call options on stock S. Thereafter, pricing algorithms for other path-dependent instruments based on the historical minimum (maximum) or time average of S are easily derived. This includes all types of barrier options on stock, futures or foreign exchange.

6.2.1 Lookback option

We consider a lookback call option on a stock S, assumed to follow a log-normal process. The call has terminal payoff $C_T = S_T - m_T$, where m_t is the historical minimum of S, cf. eq. (3.1).

Given a time step Δt, we fix two values ΔZ, ΔY:

$$\begin{aligned} \Delta Z &= \sigma\sqrt{\Delta t} \\ \Delta Y &= \Delta Z; \end{aligned}$$

and quantize S and m as follows:

$$\begin{aligned} S_j^n &= S_0 e^{j\Delta Z} \\ m_k^n &= S_0 e^{k\Delta Y}. \end{aligned} \tag{6.6}$$

The choice $\Delta Z = \Delta Y$ is justified by the fact that since m is minimum over S, it necessarily yields one of those values. Therefore, m will be optimally quantized by choosing $\Delta Y = \Delta Z$.

The correlated evolutions of S and m arise from the definition of the minimum:

$$m_{t+\Delta t} = \min(m_t, S_{t+\Delta t}). \tag{6.7}$$

Under a binomial approximation of S and m, we associate to the upward (resp. downward) transition $S_j^n \to S_{j+1}^{n+1}$ (resp. $S_j^n \to S_{j-1}^{n+1}$) in S, the transition $m_k^n \to m_{k+}^{n+1}$ (resp. $m_k^n \to m_{k-}^{n+1}$) in m. The discrete equivalent of eq. (6.7) is then:

$$\begin{aligned} m_{k+}^{n+1} &= \min(m_k^n, S_{j+1}^{n+1}) \\ m_{k-}^{n+1} &= \min(m_k^n, S_{j-1}^{n+1}), \end{aligned}$$

from which the following values $k\pm$ of $k+$ and $k-$ are found, using definition (6.6) of the quantization:

$$k\pm = \min(k, j \pm 1). \tag{6.8}$$

Taking the risk-neutral probability u from a CRR approximation of the log-normal process governing S, the lookback call price equation writes:

$$\begin{aligned} C_{j,k}^n &= uC_{j+1,k+}^{n+1} + (1-u)C_{j-1,k-}^{n+1} \\ n &= N-1, \ldots, 0 \\ j, k &= -n, \ldots, n; \end{aligned} \tag{6.9}$$

with boundary condition

$$\begin{aligned} C^N_{j,k} &= S^N_j - m^N_k = S_0 e^{j\Delta Y} - S_0 e^{k\Delta Z} \\ j,k &= -N,\ldots,N. \end{aligned}$$

The range of variations of $k\pm$ is trivially bounded, i.e. from inspection of eq. (6.8), $|k\pm| \le n$. The scheme (6.9) is therefore feasible. Using eq. (6.8) to compute $k\pm$ values, eq. (6.9) is eventually backward integrated in order to get the price $C^0_{0,0}$ of the lookback call.

The pricing algorithm of a lookback put is similar. The terminal payoff changes to $P_T = M_T - S_T$, where M_t is the historical maximum of S, cf. eq. (3.1). In this case, the quantization (6.6), and the recursion equation (6.8) still hold, with M^n_k and max substituted for m^n_k and min respectively. The price equation (6.9) becomes then:

$$\begin{aligned} P^n_{j,k} &= uP^{n+1}_{j+1,k+} + (1-u)P^{n+1}_{j-1,k-} \\ n &= N-1,\ldots,0 \\ j,k &= -n,\ldots,n; \end{aligned} \tag{6.10}$$

with boundary condition

$$\begin{aligned} P^N_{j,k} &= M^N_k - S^N_j = S_0 e^{k\Delta Y} - S_0 e^{j\Delta Z} \\ j,k &= -N,\ldots,N. \end{aligned}$$

6.2.2 Average-rate (Asian) option

We consider an average-rate (Asian) call option on a stock S assumed to follow a log-normal process. We shall develop the example of a floating-strike average-rate call, with terminal payoff $C_T = \max(0, S_T - A_T)$, where A_t is the time average S, given by eq. (3.2).

Given a time step Δt, we fix two values ΔZ, ΔY:

$$\begin{aligned} \Delta Z &= \sigma\sqrt{\Delta t} \\ \Delta Y &= \rho\Delta Z,\ \rho < 1; \end{aligned}$$

and quantize S and A as follows:

$$\begin{aligned} S^n_j &= S_0 e^{j\Delta Z} \\ A^n_k &= S_0 e^{k\Delta Y}. \end{aligned} \tag{6.11}$$

Notice that since A is an average, it does not necessarily yields one of the S values. In order to preserve accuracy, the quantization step ΔY has to be smaller than ΔZ, hence $\rho < 1$. We postpone the determination of ρ until the price equation is established.

The relation giving the correlated evolutions of A with respect to S arise from eq. (3.4):

$$A_{t+\Delta t} = \frac{(t+\Delta t)A_t + \Delta t\, S_{t+\Delta t}}{t+2\Delta t} \tag{6.12}$$

Under a binomial approximation of S and A, we associate to the upward (resp. downward) transition $S_j^n \to S_{j+1}^{n+1}$ (resp. $S_j^n \to S_{j-1}^{n+1}$) in S the transition $A_k^n \to A_{k+}^{n+1}$ (resp. $A_k^n \to A_{k-}^{n+1}$) in A. The discrete equivalent of eq. (6.12) is then, with $A_0^0 = S_0^0$:

$$\begin{aligned} A_{k+}^{n+1} &= \frac{(n+1)A_k^n + S_{j+1}^{n+1}}{n+2} \\ A_{k-}^{n+1} &= \frac{(n+1)A_k^n + S_{j-1}^{n+1}}{n+2}, \end{aligned}$$

from which the following values of $k+$ and $k-$ are found:

$$k\pm \;=\; \text{nearest}\left[\frac{\log \dfrac{(n+1)e^{k\rho\Delta Z} + e^{(j\pm1)\Delta Z}}{n+2}}{\rho\Delta Z}\right]. \tag{6.13}$$

Taking the risk-neutral probability u from a CRR approximation of the log-normal process governing S, the average-rate price equation writes:

$$\begin{aligned} C_{j,k}^n &= uC_{j+1,k+}^{n+1} + (1-u)C_{j-1,k-}^{n+1} \\ n &= N-1,\ldots,0 \\ j &= -n,\ldots,n \\ k &= -k_m(n),\ldots,k_m(n); \end{aligned} \tag{6.14}$$

with boundary condition

$$\begin{aligned} C_{j,k}^N &= S_j^N - A_k^N \qquad = S_0e^{j\Delta Z} - S_0e^{k\Delta Z} \\ j &= -N,\ldots,N \\ k &= -k_m(N),\ldots,k_m(N). \end{aligned}$$

In order for the previous scheme to be feasible, the following three points need to be addressed.

- *Bound the range of variations of k (i.e. find k_m).*
  For any time step n, the maximum value $\max_k A_k^n$ for average A cannot be greater than the maximum value $\max_j S_j^n$ for asset S, hence $A_{k_m}^n \le S_n^n$. Eq. (6.11) implies then $k_m \le n/\rho$. Thus, we let:

$$k_m \;=\; \frac{n}{\rho}.$$

- *Bound the range of variations of $k\pm$.*
  Starting from eq. (6.13), and after some algebraic manipulations, one gets $|k\pm| \le (n/\rho) + 1/\rho$, hence:

$$|k\pm| \;\le\; k_m + \frac{1}{\rho}.$$

- *Find ρ value.*
  Because of the very simple structure of eq. (6.14), the propagation of error due to the quantization (6.11) can be shown (see next section) to be less than $S_0 e^{k_m} \rho \Delta Z (e^{\rho \Delta Z} - 1)$. With $\Delta Z = \sigma \sqrt{\Delta t}$, this gives a maximum relative error $\epsilon \simeq \rho \Delta Z e^{N \sigma \sqrt{\Delta t}}$. Typically, $N = 100$, $\Delta t = 0.00274$ (1 day), $\sigma \leq 1$, hence $\epsilon \simeq 0.05\,\rho$. Thus, the value $\rho = 0.02$ ensures a 0.1 % precision. In practice, cf. sec. 8.2, the value $\rho = 0.1$ is sufficient.

7 Convergence of the FSG method

7.1 Lipschitz conditions

In the previous section, we introduced the FSG method for typical path-dependent asset pricing problems. However, the principle underlying the FSG method is very general, and can in fact be applied to arbitrary diffusion equations. We analyse below the convergence of the FSG method for diffusions equations of the type (5.1).

In order to prove the convergence of the FSG method, we will assume that the boundary condition g in problem (5.1) is K_g-Lipschitz, i.e.:

$$\forall X \in R^d, Y \in R^d, \quad |g(X) - g(Y)| \leq K_g ||X - Y||_\infty$$

with

$$||X||_\infty = \max_{i \in [1,d]} |x_i|$$

This assumption is very reasonable for option pricing applications, since most payoff functions are clearly 1-Lipschitz.

For the quantization of the PDE (5.1), we select d quantization functions whose inverse map the original space variables $x_1, \ldots, x_d$ into the transformed spaces variables $y_1, \ldots, y_d$.

$$\overline{X}(t, Y) = \left(\overline{X}_1(t, y_1), \ldots, \overline{X}_d(t, y_d)\right)$$

We assume that $\overline{X}_i$ is a strictly monotonic function of its second variable y_i, and we define its inverse by the following relation:

$$\overline{X}_i^{-1}(t, \overline{X}_i(t, y_i)) = y_i$$

Finally, we assume that $\overline{X}_i$ is K_i-Lipschitz in its second variable y_i, i.e.:

$$\forall t, y_i^1, y_i^2, \quad |\overline{X}_i(t, y_i^1) - \overline{X}_i(t, y_i^2)| \leq K_i |y_i^1 - y_i^2|$$

7.2 Quantization of time

It is a well-known consequence of the central limit theorem that the standard Brownian motion is the limit when $\Delta t \longrightarrow 0$ of the binomial distribution of step $\sqrt{\Delta t}$ (see e.g. Cox and Rubinstein (1985); Duffie (1988)). We define the k-dimensional binomial process $\tilde{W}^{\Delta t}$ as follows:

$$\forall \epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{-1, 1\}^k, \quad \mathrm{Prob}(\tilde{W}^{\Delta t}(t + \Delta t) = \tilde{W}^{\Delta t}(t) + \epsilon \sqrt{\Delta t}) = \frac{1}{2^k}$$

In light of the previous discussion, we get under suitable technical conditions (see e.g. Duffie (1988)):

$$f(t, x_1, \ldots, x_d) = \lim_{\Delta t \longrightarrow 0} \tilde{f}(t, x_1 \ldots, x_d) = E_t\left(\exp\left(-\int_0^T r(\tau, \tilde{X}(\tau))d\tau\right) g(\tilde{X}(T))\right)$$

where $\tilde{X}$ is defined by the forward stochastic difference equation:

$$\tilde{X}(t+\Delta t) = \tilde{X}(t) + M(t, \tilde{X}(t))\Delta t + V(t, \tilde{X}(t))(\tilde{W}(t+\Delta t) - \tilde{W}(t))$$

Remark: The definition of $\tilde{X}$ above corresponds to choosing an Euler scheme for the quantization of the SDE. Other schemes of higher order of convergence in Δt could be used, such as Talay-Milshtein schemes (Milshtein (1974); Talay (1984)). However, we did not explore this alternative.

From the law of iterated expectations, we see that $\tilde{f}$ satisfies the following recursive backward equation:

$$\tilde{f}(t, X) = \frac{1}{1+R(t, X)} \frac{1}{2^k} \sum_{\epsilon \in \{-1,1\}^k} \tilde{f}(t+\Delta t, X + M(t, X)\Delta t + \epsilon V(t, X)\sqrt{\Delta t}) \tag{7.1}$$

where $R(t, X)$ is defined by:

$$\frac{1}{1+R(t, X)} = \exp\left(-r(t, X)\Delta t\right)$$

7.3 Quantization of space

We select d quantization steps $\Delta Y = (\Delta y_1, \ldots, \Delta y_d)$ for the transformed space variables $y_1, \ldots, y_d$. We denote by J the d-uple of integers $J = (j_1, \ldots, j_d)$, and by $J\Delta Y$ the d-uple $J\Delta Y = (j_1 \Delta y_1, \ldots, j_d \Delta y_d)$.

For any given couple (n, J), and any $\epsilon \in \{-1, 1\}^k$, we define $J^{new}(n, J, \epsilon) = (j_1^{new}, \ldots, j_d^{new})$ by:

$$\forall i \in [1, d], \quad j_i^{new} = \text{nearest}\left[\frac{\overline{X}_i^{-1}\left((n+1)\Delta t, X^*\right)}{\Delta y_i}\right]$$

with

$$X^* = \overline{X}(n\Delta t, J\Delta Y) + M(n\Delta t, \overline{X}(n\Delta t, J\Delta Y))\Delta t + \epsilon V(n\Delta t, \overline{X}(n\Delta t, J\Delta Y))\sqrt{\Delta t}$$

In other words, we obtain J^{new} by shooting forward in time the best approximating bucket in the space variables X.

Then, we approximate equation (7.1) above by:

$$\tilde{f}^{approx}(n\Delta t, J\Delta Y) = \frac{1}{1+R(n\Delta t, \overline{X}(n\Delta t, J\Delta Y))}$$
$$\frac{1}{2^k} \textstyle\sum_{\epsilon \in \{-1,1\}^k} \tilde{f}^{approx}((n+1)\Delta t, \overline{X}((n+1)\Delta t, J^{new}(n, J, \epsilon)\Delta Y)) \tag{7.2}$$

7.4 Convergence

Combining (7.2) with (7.1) and with the Lipschitz properties of functions g and $\overline{X}$, we see that at time $(N-1)\Delta t$, we have for all possible X:

$$|\tilde{f}^{approx}((N-1)\Delta t, X) - \tilde{f}((N-1)\Delta t, X)| \leq K_g \max_{i\in[1,d]} K_i \max_{i\in[1,d]} \Delta y_i$$

Then, a straightforward backward induction on n shows that the same inequality is true for all times prior to N. Finally:

$$\forall n \leq N, \forall X, \quad |\tilde{f}^{approx}(n\Delta t, X) - \tilde{f}(n\Delta t, X)| \leq K_g \max_{i\in[1,d]} K_i \max_{i\in[1,d]} \Delta y_i$$

Hence, we can guarantee a precision on the result of Δf by choosing the space quantization steps such that:

$$\forall j \in [1, d], \quad \Delta y_j \leq \frac{\Delta f}{K_g \max_{i\in[1,d]} K_i}$$

If the payoff is that of an option, we have typically $K_g = 1$. If we furthermore choose a logarithmic quantization, i.e. if $\overline{X}_i(t, y_i) = e^{y_i}$ then K_i is simply the maximum value taken by the variable x_i at any point in time. Therefore, it is practically easy to choose the quantization steps Δy_i for reaching a prespecified desired accuracy.

Furthermore, the above inequality shows that *the FSG method is unconditionally convergent*. Indeed, the approximate value converges towards the theoretical value whenever Δt and Δy_i converge towards zero. This is true regardless of any quantitative relationships between Δt and Δy_i.

If $k = d$, i.e. if the covariance matrix Γ is regular, then finite difference methods are also convergent, and are faster than the FSG method. However, as soon as $k < d$, i.e. when the covariance matrix is degenerate, finite difference methods introduce a spurious numerical diffusion, whereas the efficiency of the FSG method is unaffected.

The FSG method therefore is an attractive solution for all degenerate numerical valuation problems. Such problems arise in many other cases than the path-dependent case.

8 Results

This section reports European and American prices found with the FSG method for different path-dependent options. Since the FSG method reduces to the classical CRR method for ordinary options, prices of at-the-money ordinary options are also given for reference.

The stock price $S_0 = 100$ and the interest rate $r = 10\%$ are held constant, while the values of the volatility σ, the maturity T, and the exercise price K vary. Maturities 3, 6, and 12 must be read in months, corresponding to 91, 182, and 364 days respectively. Both call (C) and put (P) prices are reported. When possible, we also indicate the corresponding analytic price C_a or P_a. In all cases, we have chosen the number of time steps so as to reach a 0.1% accuracy.

Ordinary option

At-the-money ordinary call option
$S_0 = 100, r = 10\%$

σ	T	European analytic C_a	European FSG ≡ CRR C
10	3	3.438	3.435
	6	5.837	5.835
	12	10.284	10.283
20	3	5.286	5.280
	6	8.262	8.258
	12	13.244	13.241
40	3	9.148	9.138
	6	13.559	13.551
	12	20.285	20.280

At-the-money ordinary put option
$S_0 = 100, r = 10\%$

σ	T	European analytic P_a	European FSG ≡ CRR P	American FSG ≡ CRR P
10	3	0.976	0.973	1.223
	6	0.973	0.971	1.450
	12	0.792	0.791	1.632
20	3	2.823	2.818	3.064
	6	3.398	3.394	3.913
	12	3.752	3.749	4.811
40	3	6.686	6.675	6.909
	6	8.694	8.687	9.205
	12	10.794	10.788	11.944

Table 1: Reference table for an at-the-money ordinary option. Time step was set to 0.5 day for all maturities. Precision for CRR prices is 0.1%.

Prices for at-the-money ordinary options are listed in table 1. The time step was chosen constant (0.5 day) for all maturities. For moderate values of the volatility ($\leq$ 50%), about $N = 100$ time steps are enough to get 0.1% precision. The time complexity is $O(N^2)$, and the memory requirement $O(N)$. The computation time for one call/put is a few milliseconds on a DEC alpha PC, and the required memory 20 Kilobytes.

8.1 Lookback option

Results for a lookback option are presented in table 2. The time step was chosen constant (1 day) for all maturities. In practice, $N = 100$ time steps are sufficient. The time complexity is $O(N^3)$, the memory requirement is $O(N^2)$. The computation time for one call/put is about 1 second on a DEC alpha PC, and the required memory 400 Kilobytes.

Lookback payoffs depend on extreme values of the underlying asset S. The continuous time framework, in which closed-form formulas are derived, captures the variations of those extrema over infinitely small time periods. In practice however, the extrema values in a lookback contract are to be computed on a daily basis. Therefore, lookback prices computed from a closed-form formula incorporate spurious intra-day variations, and thus overestimate the prices. Compared to the FSG method with a 1 day time step, this overestimation is about 5%[3].

[3] By contrast, both the CRR approximation to the ordinary option, and the FSG approximation to the average-rate with 100 time steps give European prices close to within 0.1% to their respective analytic (continuous) prices.

Lookback call option
$S_0 = 100, r = 10\%$

σ	T	European analytic C_a	European FSG C	American FSG C
10	3	5.266	5.028	5.028
	6	8.253	8.019	8.019
	12	13.298	13.075	13.075
20	3	8.939	8.482	8.482
	6	13.174	12.733	12.733
	12	19.614	19.203	19.203
40	3	16.052	15.209	15.209
	6	22.658	21.872	21.872
	12	31.804	31.106	31.106

Lookback put option
$S_0 = 100, r = 10\%$

σ	T	European analytic P_a	European FSG P	American FSG P
10	3	2.927	2.668	3.087
	6	3.632	3.368	4.209
	12	4.281	4.012	5.630
20	3	6.969	6.434	6.845
	6	9.283	8.729	9.606
	12	12.021	11.448	13.307
40	3	15.559	14.406	14.811
	6	21.685	20.456	21.362
	12	29.906	28.583	30.661

Table 2: Results of the FSG method for a lookback option. Time step was set to 1 day for all maturities. Precision for FSG prices is 0.1%. Lookback analytic prices are overestimated by about 5%.

On the other hand, we checked that the above discrepancy would vanish as the time step tends towards 0. Results are presented in table 3 below for a 3 months lookback call option. Interestingly enough, while the decrease of the precision $\Delta C/C$ is linear in the time step for an ordinary option as well as an average-rate option, it is only square root for the lookback option.

Finally, in order to estimate the accuracy of the FSG method, we have checked by an additional convergence test that 1 day time step yields a least 0.1% accuracy in the computed price.

Lookback options are more expensive than ordinary at-the-money options, roughly twice as much. As pointed out in Conze and Viswanathan (1991), the values of an American and a European lookback call are equal. Also, the price of an American lookback put is always greater than the European one.

However, table 4 shows that the upper bounds on the lookback put price derived with Snell envelopes techniques in Conze and Viswanathan (1991), are quite loose. This is especially true for short maturities, for which the price is overestimated by a factor of more than 100% on average.

8.2 Average-rate option

Results for fixed-strike and floating-strike average-rate options are presented in tables 6 and 5 respectively. Because the average variable needs a dense quantization ($\rho = 0.1$), the required

Lookback call option
$S_0 = 100$, $r = 10\%$

		European			
σ	T	analytic C_a	step (day)	FSG C	$(C_a - C)/C$
20	3	8.939	2	8.250	7.71%
			1	8.480	5.13%
			0.5	8.611	3.67%
			0.1	8.790	1.66%
			0.05	8.833	1.12%

Table 3: Measured convergence of the FSG method for a lookback option.

Lookback put option
$S_0 = 100$, $r = 10\%$

		American		
σ	T	FSG P	Snell u. b. $\bar{P}$	error $\lvert(\bar{P} - P)/P\rvert$
10	3	3.087	5.260	70 %
	6	4.209	8.653	105 %
	12	5.630	14.920	165 %
20	3	6.845	9.121	33 %
	6	9.606	14.288	49 %
	12	13.307	23.135	74 %
40	3	14.811	17.294	17 %
	6	21.362	26.614	25 %
	12	30.661	42.067	37 %

Table 4: Comparison of American lookback FSG put prices and corresponding Snell envelope upper bounds.

Floating-strike average-rate call option
$S_0 = 100$, $r = 10\%$

σ	T	European C	American C
10	3	1.852	2.142
	6	3.092	3.404
	12	5.382	5.670
20	3	2.943	3.689
	6	4.555	5.514
	12	7.259	8.463
40	3	5.195	6.817
	6	7.673	9.860
	12	11.509	14.446

Floating-strike average-rate put option
$S_0 = 100$, $r = 10\%$

σ	T	European P	American P
10	3	0.632	1.194
	6	0.671	1.494
	12	0.614	1.799
20	3	1.724	2.773
	6	2.135	3.684
	12	2.498	4.812
40	3	3.981	5.996
	6	5.260	8.223
	12	6.785	11.229

Table 5: Results of the FSG method for a floating-strike average-rate option. Time Steps were set to 1, 2, and 3 days for 3, 6, and 12 months maturities respectively. Precision for FSG prices is 0.1%.

memory is $1/\rho = 10$ times as much as in the lookback case. Maintaining a 1 day time step for all maturities would thus require too much memory on casual machines. Time Steps were set to 1, 2, and 3 days for 3, 6, and 12 months maturities respectively, so as to get about $N = 100$ time steps in each case. This is still quite reasonable. Time and space complexities are $O(N^3/\rho)$ and $O(N^2/\rho)$ respectively. The computation time for one call/put is about 15 second on a DEC alpha PC, and the required memory 4 Megabytes.

There is no analytic solution to the average-rate option pricing problem in general. However, as seen in sec. 3.3, the zero-strike case leads to the closed-form formula of eq. (3.6). Table 7 compares the analytic prices, and those computed with the FSG method in this case. The precision is 0.1% at least. Because the PDE (3.5) governing the average-rate price is homogeneous in the spot price S_0 and the strike K, the convergence of the FSG method does not depend on the value of K. Therefore, given the quantization with $\rho = 0.1$ of A, we can state an overall precision of at least 0.1% in our results.

Average-rate options are much cheaper than ordinary and lookback options. Also, since the average is not a strictly decreasing (increasing) time process as is the case for the minimum (maximum), American average-rate calls are worth exercising.

A possible method for approximating European call and put prices C_{arith} and P_{arith} of a fixed-strike average-rate option consists in replacing the arithmetic time-average with the geometric time-average, in which case closed-form formulas can be derived for prices C_{geo} and P_{geo} (Kemna and Vorst (1990)). Since the geometric average is always less than the arithmetic average, the following relationships holds: $C_{geo} < C_{arith}$, and $P_{geo} > P_{arith}$.

Fixed-strike average-rate call option $S_0 = 100$, $r = 10\%$				
			European	American
σ	T	K	C	
10	3	95	6.132	6.546
		100	1.869	1.967
		105	0.151	0.152
	6	95	7.248	7.632
		100	3.100	3.212
		105	0.727	0.735
	12	95	9.313	9.616
		100	5.279	5.394
		105	2.313	2.336
20	3	95	6.500	7.371
		100	2.960	3.219
		105	0.966	1.001
	6	95	7.793	8.805
		100	4.548	4.893
		105	2.241	2.337
	12	95	10.336	11.218
		100	7.079	7.521
		105	4.539	4.729
40	3	95	8.151	9.447
		100	5.218	5.826
		105	3.106	3.347
	6	95	10.425	10.927
		100	7.650	8.519
		105	5.444	5.913
	12	95	13.825	15.649
		100	11.213	12.439
		105	8.989	9.790

Fixed-strike average-rate put option $S_0 = 100$, $r = 10\%$				
			European	American
σ	T	K	P	
10	3	95	0.013	0.013
		100	0.626	0.832
		105	3.785	5.337
	6	95	0.046	0.051
		100	0.655	0.978
		105	3.039	5.287
	12	95	0.084	0.104
		100	0.577	1.079
		105	2.137	5.230
20	3	95	0.379	0.407
		100	1.716	2.066
		105	4.598	6.108
	6	95	0.731	0.820
		100	2.102	2.629
		105	4.552	6.338
	12	95	1.099	1.318
		100	2.369	3.181
		105	4.356	6.596
40	3	95	2.025	2.223
		100	3.970	4.581
		105	6.735	8.168
	6	95	3.215	3.610
		100	5.197	6.078
		105	7.748	9.438
	12	95	4.550	5.263
		100	6.465	7.761
		105	8.767	10.927

Table 6: Results of the FSG method for a fixed-strike average-rate option. Time Steps were set to 1, 2, and 3 days for 3, 6, and 12 months maturities respectively. Precision for all prices is 0.1%.

Zero-strike average-rate call option $S_0 = 100, r = 10\%$

		European		
σ	T	analytic C_a	FSG C	$\|(C_a - C)/C\|$
10	3	98.763	98.780	0.02%
	6	97.547	97.579	0.03%
	12	95.175	95.232	0.05%
20	3	98.763	98.781	0.02%
	6	97.547	97.580	0.03%
	12	95.175	95.239	0.07%
40	3	98.763	98.785	0.02%
	6	97.547	97.587	0.04%
	12	95.175	95.277	0.10%

Table 7: Accuracy of the FSG method measured against the zero-strike average-rate option. Time Steps were set to 1, 2, and 3 days for 3, 6, and 12 months maturities respectively.

Table 8 shows that the geometric average approximation is good for low terms and low volatilities. However, the error rapidly grows with the option term and volatility, and stops being acceptable (up to 10%) for one year and longer standing options.

9 Conclusion

In this paper, we have analyzed the problem of pricing path-dependent contingent claims. We have shown that these problems lead to solving degenerate diffusion PDE in the space augmented with the path-dependent variables. We have established necessary and sufficient conditions under which these degenerate PDE are holonomic, i.e. can be reduced to lower dimensional non-degenerate PDE. We have applied these results to popular types of path-dependent options. In particular, we have shown that the average-rate option pricing problem is non-holonomic.

Then, we have described a new numerical technique called the Forward Shooting Grid method (FSG) for pricing both European and American non-holonomic contingent claims, and have tested it on lookback and average-rate options. It is straightforward to use the FSG method in order to price other popular path-dependent options, such as capped or barrier options.

The FSG method proves to be as accurate as Monte Carlo simulation, with faster execution time. It is also the first method capable of dealing with the early exercise condition of American path-dependent options, showing that the Snell envelope upper bounds obtained on American

Geometric average-rate call option $S_0 = 100, r = 10\%$					Geometric average-rate put option $S_0 = 100, r = 10\%$				
			European					European	
σ	T	K	analytic	error w.r.t. FSG arith	σ	T	K	analytic	error w.r.t. FSG arith
10	3	95	6.093	0.63 %	10	3	95	0.013	-0.00 %
		100	1.830	2.08 %			100	0.627	-0.16 %
		105	0.141	6.62 %			105	3.815	-0.80 %
	6	95	7.165	1.14 %		6	95	0.047	-2.17 %
		100	3.022	2.52 %			100	0.660	-0.76 %
		105	0.683	6.05 %			105	3.079	-1.32 %
	12	95	9.160	1.64 %		12	95	0.086	-2.38 %
		100	5.135	2.73 %			100	0.587	-1.73 %
		105	2.196	5.06 %			105	2.173	-1.68 %
20	3	95	6.402	1.51 %	20	3	95	0.384	-1.32 %
		100	2.872	2.97 %			100	1.730	-0.82 %
		105	0.908	6.00 %			105	4.643	-0.98 %
	6	95	7.742	0.65 %		6	95	0.746	-2.05 %
		100	4.374	3.83 %			100	2.134	-1.52 %
		105	2.102	6.20 %			105	4.619	-1.47 %
	12	95	9.979	3.45 %		12	95	1.142	-3.91 %
		100	6.751	4.63 %			100	2.440	-3.00 %
		105	4.251	6.34 %			105	4.465	-2.50 %
40	3	95	7.862	3.54 %	40	3	95	2.089	-3.16 %
		100	4.959	4.96 %			100	4.064	-2.37 %
		105	2.889	6.99 %			105	6.870	-2.00 %
	6	95	9.867	5.35 %		6	95	3.355	-4.35 %
		100	7.136	6.72 %			100	5.380	-3.52 %
		105	4.983	8.47 %			105	7.985	-3.06 %
	12	95	12.743	7.83 %		12	95	4.847	-6.53 %
		100	10.200	9.03 %			100	6.829	-5.63 %
		105	8.048	10.47 %			105	9.203	-4.97 %

Table 8: Comparison between the European prices of a fixed-strike *geometric* average-rate call and a fixed-striked (arithmetic) average-rate call. FSG prices, which are not reported here, were taken from table 6 in order to compute the relative error.

lookback prices are quite overestimated. Our numerical experiments have also shown that the usual geometric average approximation of arithmetic average-rate options is inaccurate.

The FSG method is a general purpose solution technique for arbitrary multidimensional advection-diffusion equations. However, since these problems require a memory space exponential in the number of variables, the FSG method, like any finite difference or lattice-based method, can only be used on problems with few variables.

Unlike Finite Difference Methods, the FSG method is unconditionally convergent, even when the diffusion term is degenerate. Such degeneracy arises in several other important pricing problems. In particular, we plan to investigate the application of the FSG method to the pricing of path-dependent interest rate contingent claims such as Mortgage-Backed securities.

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