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Elementary Divisors of Normal Matrices

It is known from the transformation theory of matrices that normal matrices can be brought to the diagonal form by means of a similarity transformation. On the other hand, any diagonal matrix possesses linear elementary divisors exclusively. Since elementary divisors are unaltered by similarity transformations, it follows that normal matrices have linear elementary divisors only.

A direct proof of this fact is given below. It is an extension of the proof of Wedderburn, which demonstrates the same fact for Hermitian matrices.[1]

Let N be a normal matrix of degree n over a generalized complex field,† i.e., $NN^* = N^*N$ (the star denotes the complex-conjugated and transposed matrix); or, if A, B denote the Hermitian components of N, where

$$N = A + iB \,, \tag{1}$$

the normality is expressed by the equation

$$AB = BA \,. \tag{2}$$

Let $m(z)$ denote the minimum polynomial of N, i.e., the polynomial of lowest degree for which $m(N) = 0$.

Suppose N possesses a nonlinear elementary divisor. Then $m(z)$ possesses a multiple root z_k, because $m(z)$ is the ratio of the determinant $f(z) = |zE - N|$ and the greatest common divisor of all minors of $f(z)$ with degree $n-1$. (Cf. Ref. 2). Thus

$$m(z) = (z - z_k)^{\alpha_k} p(z) \qquad (\alpha_k > 1) \,. \tag{3}$$

We introduce

$$m_1(z) = (z - z_k)^{\alpha_k - 1} p(z) \tag{4}$$

and write $m_1(N)$ as the sum of its hermitian components

$$m_1(N) = C + iD \,. \tag{5}$$

Then

$$CD = DC \,, \tag{6}$$

because any polynomial of a normal matrix is itself normal.

Now $[m_1(z)]^2$ is a multiple of $m(z)$, and $m(N) = 0$. Hence

$$0 = [m_1(N)]^2 = C^2 - D^2 + iCD + iDC$$
$$= C^2 - D^2 + 2iCD \,. \tag{7}$$

This equation can be satisfied only if

$$CD = 0 \qquad \text{and} \qquad C^2 - D^2 = 0 \,. \tag{8}$$

Thus

$$(CD)^2 = C^2D^2 = C^4 = D^4 = 0 \,. \tag{9}$$

C and D being hermitian matrices, $C = D = 0$ follows from the vanishing of an even power of C and D, as $D^2 = 0$ implies

$$trD^2 = \sum_i \sum_k d_{ik}\bar{d}_{ki} = \sum_{ik} |d_{ik}|^2 = 0 \,. \tag{10}$$

Hence

$$m_1(N) = 0 \tag{11}$$

holds true.

The degree of $m_1(z)$ is lower by unity than the degree of $m(z)$, whereas $m(z)$ is by definition the polynomial of lowest degree possessing N as a root. This contradiction forces us to renounce the supposition of the existence of a nonlinear elementary divisor.

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†A generalized complex field is defined by extending an arbitrary ordered commutative field K with characteristic $\neq 2$ by adjoining a square root ξ of a negative number. Complex conjugation is then defined by the automorphism in $K(\xi)$ induced by $\xi \rightarrow -\xi$. The proof of the text, which for the sake of simplicity is written down for $K(\xi) =$ field of ordinary complex numbers, also holds true for generalized complex fields. In that case, i should be replaced by ξ, and, if the field $K(\xi)$ is not algebraically complete, $(z - z_k)$ should be replaced by p_k, p_k being a prime of the polynomial principal ideal ring $K(\xi)[z]$.

The assumption: characteristic $\neq 2$ is used in defining the Hermitian components of a matrix. On the other hand $\xi^2 < 0$ assures $d_{ik}\bar{d}_{ik} > 0$, a necessary condition to imply $D = 0$ if $D^2 = 0$.

References

1. J. H. M. Wedderburn, *Ann. of Math.* **27,** 245 (1926), and p. 89, Ref. (2).
2. J. H. M. Wedderburn, *Lectures on Matrices,* American Mathematical Society Coll. Publ. 17, New York, p. 23, 1934.

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