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Theory on the Speed of Convergence in Adaptive Equalizers for Digital Communication*

Abstract: This paper presents an analysis of the convergence properties of adaptive transversal equalizers minimizing mean-square distortion. The intention is to reveal the influence on the speed of convergence exerted by the number of taps, the step-size parameter in the adjustment loops, and the spectrum of the unequalized signal. Attention is focused on the convergence of the expected mean-square distortion. Several approximations are made in the analysis, among them the approximation of higher-order statistics by second-order statistical parameters. Comparison with results obtained by computer simulation, however, shows that the theory developed renders a quite accurate picture of the convergence process.

Previous work in this field demonstrated the limits set to the speed of convergence by the extreme values of the power spectrum of the unequalized signal. It is shown here that, with regard to the mean-square distortion, the influence of the number of taps will usually dominate by far. The theory provides a simple criterion for convergence and answers the question of how to attain the fastest convergence.

Introduction

Synchronous data transmission over the existing telephone network at speeds of several kbps requires equalization. Most equalizers presently used in modem receivers are of the transversal filter type. Various methods for adjusting their tap gains automatically have been described in the literature [1–9]. Basically there are two kinds of automatic adjustment processes. The first involves sending a series of isolated test pulses prior to data transmission. The equalizer settings derived in this initial "training" period are kept constant during the subsequent period of data transmission. The second is known as *adaptive equalization.* Here the equalizer settings are directly derived from the received data signal. Adaptive equalizers seek continuously to minimize the deviation of their sampled output signal from a quantized reference signal that resembles the transmitted pulse amplitudes. During the initial training period an ideal reference signal can be made available to the equalizer by transmitting a known sequence of pseudorandom data over the channel and by generating internally in the receiver an identical sequence in proper synchronism. When actual data are transmitted, the residual distortion has usually decreased to a small value. The equalizer can then use the reconstructed output signal of the receiver as a reference signal. In this so-called *decision-directed mode* the effect of false decisions is usually negligible. Thus the adaptation mechanism continues to be effective during the entire period that data are transmitted.

In many practical applications the transmitted messages are short. The start-up time (during which the receiver locks on to the carrier, establishes bit synchronization, and performs automatic equalization) may constitute a substantial portion of the total holding time. Particularly in multiparty polling systems, the start-up time of the individual receivers can seriously affect the line utilization. For the early automatic equalizers, settling times in the order of seconds have been reported. In the meanwhile great improvements have been achieved.

In this paper we shall present a theory on the convergence process in adaptive equalizers which employ the mean-square (MS) algorithm [3–7]. The aim of the analysis is to reveal the influence of various system parameters on the convergence process and thus to establish the limits of the speed of convergence.

In a similar investigation Gersho [5] considered the expected tap-gain errors relative to their optimum settings. He showed that the optimum speed at which these expectations may converge to zero is largely determined by the maximum and minimum values of the power density spectrum of the unequalized signal. Gersho's analysis suggests that the number of taps apparently has little influence on the convergence process. Similar results, but not for exactly the same equalizer, have been reported by Chang [8] and Kobayashi [9]. Gersho also considered the expected variance of the tap-gain errors and presented a general proof of convergence. Here we shall extend Gersho's analysis. Con-

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sidering the mean-square distortion, we show that the number of taps is generally much more important than the extreme values of the power density spectrum. A criterion for stability is derived and the question of how to attain fastest convergence is addressed.

We first review the fundamentals of this type of equalizer. Later on, a theory on the convergence of the expected mean-square distortion is developed. The theoretical results are then compared with results obtained by computer simulation.

Review of fundamentals

The equalizer considered in this paper is depicted in Fig. 1. It consists basically of a linear transversal filter that transforms the input signal $x(t)$ into the output signal $z(t)$. The equalized signal is sampled at regular intervals T. As the input signal we consider a PAM base-band signal

$$x(t) = \sum_{n=-\infty}^{+\infty} a_n h(t-nT) + w(t) , \tag{1}$$

where the $\{a_n\}$ are quantized pulse amplitudes, $h(t)$ is the channel response, T the baud interval, and $w(t)$ the additive noise. Let $\mathbf{c}_n$ be the vector of tap gains of the transversal filter, and $\mathbf{x}_n$ the vector of tap output signals, both at the nth sampling instant. These vectors are N-dimensional, N being the number of taps. Throughout the paper a prime (′) denotes transposition. At the nth sampling instant the output signal of the equalizer reads $z_n = \mathbf{c}_n'\mathbf{x}_n$. It deviates from the originally transmitted pulse amplitude by $e_n = z_n - a_n$. In this study we adopt the familiar mean-square distortion criterion. For a particular $\mathbf{c}_n$ the mean-square distortion is defined as the average value of e_n^2 over all possible pulse amplitude and noise sequences [3-9]:

$$\langle e_n^2 \rangle = \langle (\mathbf{c}_n'\mathbf{x}_n - a_n)^2 \rangle = \mathbf{c}_n'\mathbf{R}\mathbf{c}_n - 2\mathbf{c}_n'\mathbf{b} + \langle a_n^2 \rangle . \tag{2}$$

Here $\mathbf{R}$ is a positive definite $N \times N$ matrix with elements

$$r_{ik} = \langle x_{ni} \cdot x_{nk} \rangle, \qquad 1 \le i,k \le N , \tag{3}$$

and $\mathbf{b}$ denotes an N-dimensional vector with elements

$$b_i = \langle a_n \cdot x_{ni} \rangle, \qquad 1 \le i \le N . \tag{4}$$

$\mathbf{R}$ and $\mathbf{b}$ do not depend on n since we assume that both the sequence $\{a_n\}$ and the noise $w(t)$ are stationary.

The mean-square distortion assumes its minimum value, $\langle e_{\text{opt}}^2 \rangle$, when $\mathbf{c}_n$ is chosen equal to

$$\mathbf{c}_{\text{opt}} = \mathbf{R}^{-1}\mathbf{b} . \tag{5}$$

Introducing a tap-gain error vector

$$\mathbf{p}_n = \mathbf{c}_n - \mathbf{c}_{\text{opt}} \tag{6}$$

we have

$$\langle e_n^2 \rangle = \mathbf{p}_n'\mathbf{R}\mathbf{p}_n + \langle e_{\text{opt}}^2 \rangle . \tag{7}$$

The MS algorithm [3-7], the convergence properties of which will be investigated in this paper, is intended to minimize the mean-square distortion. Note that this is equivalent to minimizing the positive definite quadratic term $\mathbf{p}_n'\mathbf{R}\mathbf{p}_n$ ("excess mean-square distortion"). New values are assigned to $\mathbf{p}_n$, and hence to $\mathbf{c}_n$, at each sampling instant by estimating the gradient $\partial\langle e_n^2\rangle/\partial\mathbf{p}_n$ and modifying $\mathbf{p}_n$ accordingly. The MS algorithm results from taking $\partial e_n^2/\partial\mathbf{p}_n = 2\, e_n\mathbf{x}_n$ as an unbiased estimate of $\partial\langle e_n^2\rangle/\partial\mathbf{p}_n$. This leads, then, to the iterative formula

$$\mathbf{p}_{n+1} = \mathbf{p}_n - \alpha_{(n)} e_n \mathbf{x}_n . \tag{8}$$

Defining $e_{n\text{opt}} = \mathbf{c}_{\text{opt}}'\mathbf{x}_n - a_n$ we have

$$e_n = e_{n\text{opt}} + \mathbf{p}_n'\mathbf{x}_n . \tag{9}$$

Note that

$$\langle e_{n\text{opt}} \cdot \mathbf{x}_n \rangle = \mathbf{0} . \tag{10}$$

Fig. 1 shows the implementation of the algorithm.

The step-size parameter $\alpha_{(n)}$ may vary as a function of time. To begin with we shall consider a constant step-size parameter. For $\alpha > 0$ and sufficiently small, $\mathbf{p}_n$ converges in the mean towards $\mathbf{0}$ from arbitrary initial settings $\mathbf{p}_0$ because of the positive definite quadratic nature of the excess mean-square distortion. Noise and the finiteness of the number of taps make it impossible to attain zero mean-square distortion. Thus, in the steady state, finite output errors e_n cause $\mathbf{p}_n$ to fluctuate randomly about $\mathbf{0}$—with zero mean, but finite variance.

Our major concern in the analysis will be devoted to the speed of convergence of the expected mean-square distortion, denoted by $E(\langle e_n^2\rangle)$. This quantity represents the ensemble mean value of $\langle e_n^2\rangle$, subject to averaging $\langle e_n^2\rangle$ over $\mathbf{p}_n$. $E(\langle e_n^2\rangle)$ converges towards $\langle e_{\text{opt}}^2\rangle$ as $\mathbf{p}_n$ converges towards $\mathbf{0}$, but because of the finite variance of $\mathbf{p}_n$ in the steady state, it will settle at a value greater than $\langle e_{\text{opt}}^2\rangle$.

Analysis of the convergence process

We shall first introduce a transformation that considerably facilitates further analysis. The convergence properties of $E(\mathbf{p}_n)$ can then easily be examined. In the remainder of this section a theory on the convergence of $E(\langle e_n^2\rangle)$ will be developed.

• *Coordinate transformation*

Since $\mathbf{R}$ is symmetric it can be represented in the form

$$\mathbf{R} = \mathbf{U}\,\text{Diag}\,(\boldsymbol{\rho})\,\mathbf{U}' . \tag{11}$$

($\boldsymbol{\rho}$ is the vector of the eigenvalues of $\mathbf{R}$; $\rho_i > 0$, $1 \le i \le N$, since $\mathbf{R}$ is positive definite; $\mathbf{U}$ is the unitary matrix whose

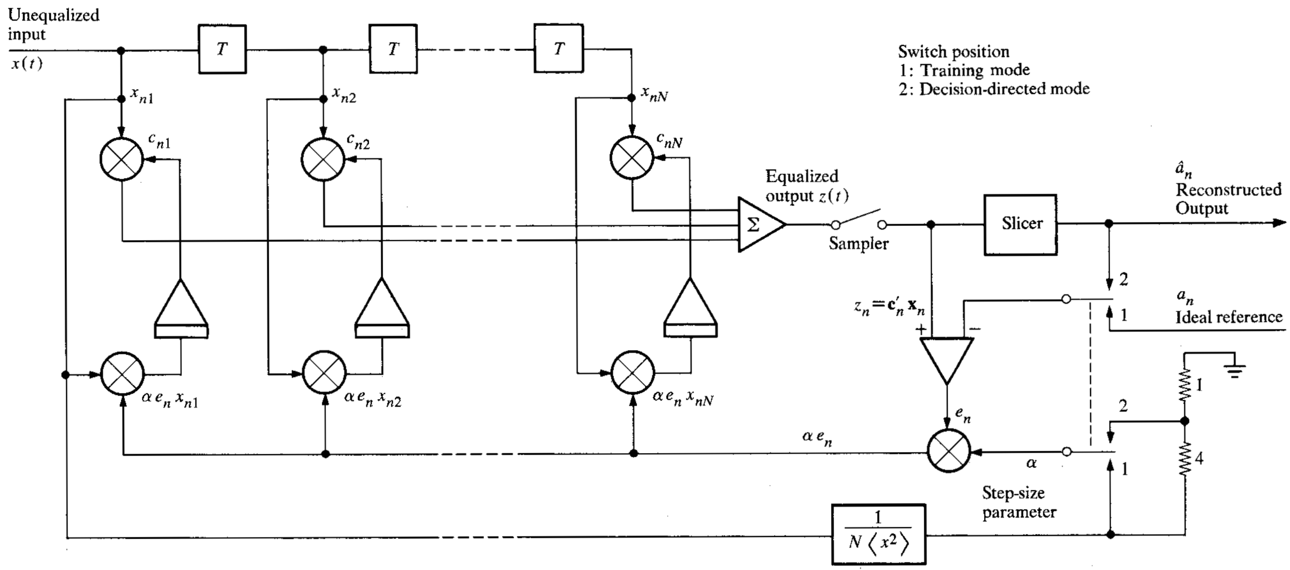


Figure 1 Adaptive transversal equalizer employing the MS algorithm with controlled step-size parameter.

ith column is the eigenvector $\mathbf{u}_i$ of $\mathbf{R}$, associated with ρ_i.) We now introduce

$$\mathbf{y}_n = \mathbf{U}'\mathbf{x}_n, \tag{12}$$

and

$$\mathbf{q}_n = \mathbf{U}'\mathbf{p}_n. \tag{13}$$

This transformation is equivalent to a rotation of the coordinate system. The elements of the modified tap-gain vector $\mathbf{y}_n$ are uncorrelated:

$$\langle y_{ni} \cdot y_{nk} \rangle = 0, \qquad i \neq k$$
$$= \rho_i, \qquad i = k. \tag{14}$$

Multiplication of (8) by $\mathbf{U}'$ from the left yields

$$\mathbf{q}_{n+1} = \mathbf{q}_n - \alpha e_n \mathbf{y}_n. \tag{15}$$

Similarly, we obtain from (9) and (10),

$$e_n = e_{n\text{opt}} + \mathbf{q}_n'\mathbf{y}_n \tag{16}$$

and

$$\langle e_{n\text{opt}} \cdot \mathbf{y}_n \rangle = \mathbf{0}. \tag{17}$$

• *Convergence properties of* $E(\mathbf{p}_n)$ *and* $E(\mathbf{q}_n)$

From (15), (16), and (17) it follows that

$$E(\mathbf{q}_{n+1}) = E(\mathbf{q}_n) - \alpha\, E[(\mathbf{q}_n'\mathbf{y}_n)\, \mathbf{y}_n]. \tag{18}$$

In order to facilitate further mathematical treatment, $\mathbf{p}_n$ and $\mathbf{x}_n$ are assumed to be statistically independent of each other. The same applies then to $\mathbf{q}_n$ and $\mathbf{y}_n$ and thus $E(\mathbf{q}_n)$ can be extracted from the rightmost term in (18). Since $\mathbf{p}_n$ depends on $\mathbf{x}_{n-1}, \mathbf{x}_{n-2}, \cdots$, and $\mathbf{x}_n$ merely comprises the tap output signals after another baud interval ($x_{n(i+1)} = x_{(n-1)i}$, $1 \le i < N$), the assumption is not strictly true. In view of small step-size parameters, however, Gersho [5] felt that the dependence between $\mathbf{p}_n$ and $\mathbf{x}_n$ is weak and can therefore be neglected. With this assumption, (18) assumes the form

$$E(\mathbf{q}_{n+1}) = \text{Diag}\ (\mathbf{1} - \alpha\boldsymbol{\rho})\ E(\mathbf{q}_n). \tag{19}$$

It should be mentioned here that in a recent paper [11] an attempt was made to include the dependency between $\mathbf{p}_n$ and $\mathbf{x}_n$ in the analysis.

Several authors [5,9] have shown that the eigenvalues of $\mathbf{R}$ are bounded by

$$\frac{1}{T}\ \text{Inf}\ P^*(\omega) < \rho_i < \frac{1}{T}\ \text{Sup}\ P^*(\omega), \qquad 1 \le i \le N, \tag{20}$$

where $P^*(\omega)$ represents the periodic power density spectrum of the sampled unequalized signal: $P^*(\omega) = P^*[\omega + (2\pi/T)]$. The extreme eigenvalues approach the bounds as N goes to infinity.

Let $\rho_{\min}$ and $\rho_{\max}$ denote the smallest and the largest eigenvalues of $\mathbf{R}$. From (19) it follows that $E(\mathbf{q}_n)$ converges to $\mathbf{0}$ if

$$0 < \alpha < 2/\rho_{\max}. \tag{21}$$

Because of (13), the same applies to $E(\mathbf{p}_n)$. Gersho [5] has shown that for $\alpha = 2/(\rho_{\min} + \rho_{\max})$ fastest convergence takes place. The Euclidian norm of $E(\mathbf{q}_n)$, which is equal to the Euclidian norm of $E(\mathbf{p}_n)$, is then reduced at least by the factor $(\rho_{\max} - \rho_{\min})/(\rho_{\max} + \rho_{\min})$ in each iteration. Since $\rho_{\min}$ and $\rho_{\max}$ resemble in good approximation the extreme values of $P^*(\omega)$, a direct relationship between $P^*(\omega)$ and the optimum speed of convergence of $E(\mathbf{q}_n)$ results.

One might suspect that $\rho_{\min}$ and $\rho_{\max}$ determine to a similar extent also the optimum speed of convergence of the expected mean-square distortion, $E(\langle e_n^2\rangle)$. Later in this paper, however, we shall see that the convergence properties of $E(\langle e_n^2\rangle)$ depend, unlike $E(\mathbf{q}_n)$, also on the number of taps, N. In fact, for most practical cases N will be the dominating factor which, for convergence of $E(\langle e_n^2\rangle)$, imposes a condition much tighter than (21) on the values of α. There exist values of α for which $E(\mathbf{q}_n)$ converges, but $E(\langle e_n^2\rangle)$ diverges. This property suggests that, if both quantities converge, $E(\mathbf{q}_n)$ will generally converge much faster than $E(\langle e_n^2\rangle)$. In the following analysis we may therefore assume that $E(\mathbf{q}_n)$ becomes rapidly negligible during the equalization process if it were not already zero from the beginning, i.e.,

$$E(\mathbf{q}_n) \approx \mathbf{0}. \qquad (22)$$

• *Convergence properties of* $E(\langle e_n^2\rangle)$

Equation (13) enables us to decompose $\mathbf{p}_n'\mathbf{R}\mathbf{p}_n$ into N components. Actually we are interested in the expectation thereof:

$$E(\mathbf{p}_n'\mathbf{R}\mathbf{p}_n) = \sum_{i=1}^{N}\rho_i E(q_{ni}^2) = \boldsymbol{\rho}'\mathbf{s}_n. \qquad (23)$$

Using (22) and the assumption that $\mathbf{q}_n$ and $\mathbf{y}_n$ were statistically independent of each other, and making some further approximations, we show in the Appendix that

$$\mathbf{s}_{n+1} \simeq \mathbf{A}\mathbf{s}_n + \alpha^2 \langle e_{\text{opt}}^2\rangle \boldsymbol{\rho}, \qquad (24)$$

where

$$\mathbf{A} = \begin{bmatrix} (1-\alpha\rho_1)^2 & \alpha^2\rho_1\rho_2 & \cdots & \alpha^2\rho_1\rho_N \\ \alpha^2\rho_2\rho_1 & (1-\alpha\rho_2)^2 & \cdots & \alpha^2\rho_2\rho_N \\ \vdots & & & \vdots \\ \alpha^2\rho_N\rho_1 & & & (1-\alpha\rho_N)^2 \end{bmatrix} \qquad (25)$$

The matrix $\mathbf{A}$ is symmetric and its elements are all positive. The matrix, however, is not necessarily positive definite. Similarly to (11), we introduce

$$\mathbf{A} = \mathbf{V}\ \text{Diag}\ (\boldsymbol{\lambda})\ \mathbf{V}'. \qquad (26)$$

($\boldsymbol{\lambda}$ is the vector of the eigenvalues of $\mathbf{A}$; $\mathbf{V}$ is the unitary matrix whose ith column is the eigenvector $\mathbf{v}_i$ of $\mathbf{A}$, associated with λ_i.) Let $\langle x^2\rangle$ denote the mean-square value of the tap output signals. Using the relation

$$N\langle x^2\rangle = \text{trace } \mathbf{R} = \sum_{i=1}^{N}\rho_i, \qquad (27)$$

it can be shown that the solution of (24) reads

$$\mathbf{s}_n \simeq \sum_{i=1}^{N}\gamma_i\lambda_i^n\mathbf{v}_i + \frac{\alpha\langle e_{\text{opt}}^2\rangle}{(2-\alpha N\langle x^2\rangle)}\mathbf{1}, \qquad (28)$$

where γ_i is determined by the initial conditions

$$\gamma_i = \mathbf{v}_i'\left(\mathbf{s}_0 - \frac{\alpha\langle e_{\text{opt}}^2\rangle}{(2-\alpha N\langle x^2\rangle)}\mathbf{1}\right), \qquad 1 \le i \le N,$$

$$s_{0i} = E(q_{0i}^2) = E[(\mathbf{u}_i'\mathbf{p}_0)^2], \qquad 1 \le i \le N.$$

Substituting (28) into (23) and observing (7) we finally obtain

$$E(\langle e_n^2\rangle) \simeq \sum_{i=1}^{N}\delta_i\lambda_i^n + \frac{2\langle e_{\text{opt}}^2\rangle}{(2-\alpha N\langle x^2\rangle)}, \qquad (29)$$

where

$$\delta_i = (\mathbf{v}_i'\boldsymbol{\rho})\cdot\gamma_i. \qquad (30)$$

In (29) the first term on the right-hand side describes the transient behavior of $E(\langle e_n^2\rangle)$, whereas the second term represents its steady-state value.

• *Transient behavior of* $E(\langle e_n^2\rangle)$

In investigating the properties of $\mathbf{A}$ we shall be able to make some observations concerning the transient behavior of $E(\langle e_n^2\rangle)$. Among them we present a new criterion for stability. We also indicate that a spread of the eigenvalues of $\mathbf{R}$ has not the strong influence on the speed of convergence of $E(\langle e_n^2\rangle)$ that might be expected from considering the convergence properties of $E(\mathbf{q}_n)$. Furthermore, for all eigenvalues of $\mathbf{R}$ being equal we show that only the largest eigenvalue of $\mathbf{A}$ determines the speed of convergence of $E(\langle e_n^2\rangle)$:

a) All eigenvalues of $\mathbf{A}$ are real numbers since $\mathbf{A}$ is symmetric. Hence, the transient of $E(\langle e_n^2\rangle)$ will exhibit no oscillations.
b) For $\alpha \to 0$ all eigenvalues of $\mathbf{A}$ approach unity.
c) The equalizer is stable and the expected mean-square distortion converges to a steady state if $|\lambda_i| < 1$, $1 \le i \le N$. This will be the case if α satisfies

$$0 < \alpha < 2/N\langle x^2\rangle = 2/\sum_{i=1}^{N}\rho_i. \qquad (31)$$

Proof: The N elements of the ith row of $\mathbf{A}$ add up to

$$\sum_{k=1}^{N}a_{ik} = 1 - \alpha\rho_i(2 - \alpha N\langle x^2\rangle).$$

If α satisfies (31), then each row sum of $\mathbf{A}$ is smaller than unity. A matrix which has this property and whose elements are all positive can have only eigenvalues with absolute value smaller than unity [10].

The criterion for stability thus found imposes a much narrower upper bound on α than (21). It clearly exhibits the significance of the number of taps while showing no dependence on the distribution of the eigenvalues of **R**. We recall that fastest convergence of $E(\mathbf{q}_n)$ takes place for $\alpha = 2/(\rho_{\min} + \rho_{\max})$. The new criterion indicates that with this step-size parameter $E(\langle e_n^2 \rangle)$ would diverge, provided $N > 2$. Since $E(\langle e_n^2 \rangle)$ is closely related to the probability of errors, (31) must be considered as a necessary condition. Intuitively this dependence on the number of taps could be expected, since for a given step-size parameter each additional tap increases through its tap-gain fluctuations the expected excess mean-square distortion, $E(\mathbf{p}_n' \mathbf{R} \mathbf{p}_n)$. Expansion of the number of taps without decreasing the step-size parameter must therefore lead to instability.

d) A small eigenvalue of **R** ($\rho_i \to 0$) leads to a slowly converging term in (29), ($\lambda_i \to 1$). But the slower the term converges relative to the other terms, the smaller the probability that this term contributes significantly to $E(\langle e_n^2 \rangle)$, ($\delta_i \to 0$).

Proof: For $\rho_i = 0$ the ith row of **A** reads $\{0,\cdots,0, a_{ii} = 1,\cdots,0\}$. Consequently, $\lambda_i = 1$ and $\mathbf{v}_i' = \{0,\cdots,0,y_{ii} = 1,0,\cdots,0\}$. Since $\mathbf{v}_i' \boldsymbol{\rho} = 0$ it follows from (30) that $\delta_i = 0$.

Generally, it is true that a larger spread of the eigenvalues of **R** leads to slower convergence. But the fact that the slower-converging terms in (29) are usually given smaller weights acts to alleviate the effect. Thus a spread of the eigenvalues of **R** affects the convergence of $E(\langle e_n^2 \rangle)$ less than the convergence of $E(\mathbf{q}_n)$.

e) For all eigenvalues of **R** being equal, i.e., $\rho_i = \langle x^2 \rangle$, $1 \le i \le N$, the largest eigenvalue of **A** is given by

$$\lambda_{i\max} = 1 - \alpha \langle x^2 \rangle (2 - \alpha N \langle x^2 \rangle) . \qquad (32)$$

The other eigenvalues of **A** have no influence on the transient behavior of $E(\langle e_n^2 \rangle)$ since $\delta_i = 0$, $i \ne i_{\max}$.

Proof: It can easily be verified that $\lambda_{i\max}$ is an eigenvalue of **A** and that $\mathbf{v}_{i\max}' = N^{-1/2} \{1,1,\cdots,1\}$ represents the associated eigenvector. It follows from the Perron-Frobenius theorem [10] on positive matrices that $\lambda_{i\max}$ is indeed the largest eigenvalue of the positive matrix **A**. The theorem says that the largest eigenvalue of a positive matrix is a positive real number and the associated eigenvector consists entirely of positive elements. Because the eigenvectors of **A** form a set of orthogonal vectors, only one eigenvector can have this property. Since $\mathbf{v}_{i\max}$ consists entirely of positive elements, $\lambda_{i\max}$ must be the largest eigenvalue of **A**. Since $\mathbf{v}_{i\max}$ is parallel to $\boldsymbol{\rho}$, the other eigenvectors of **A** are orthogonal to $\boldsymbol{\rho}$. Consequently, $\delta_i = 0$, $i \ne i_{\max}$.

• *The steady state*

The rightmost term of (29) reveals a simple relationship between α and the steady-state value of $E(\langle e_n^2 \rangle)$. We again see that a steady state exists only if α, satisfies (31). Equation (28) indicated that in the steady state all elements of $\mathbf{s}_n$ become equal. Taking into account $E(q_{ni} q_{nk}) \to 0$, $i \ne k$, as shown in the Appendix, we find that in the steady state the tap gains fluctuate with equal variance but in an uncorrelated fashion about their optimum settings.

• *Optimum speed of convergence for all eigenvalues of* **R** *being equal*

Data communication over telephone channels suffers generally more from phase distortion than from amplitude distortion. If the modulation scheme provides for a flat amplitude characteristic and the spectrum of the transmitted signal is not shaped by coding techniques, the eigenvalues of **R** will be clustered closely about $\langle x^2 \rangle$. Let us assume $\rho_i = \langle x^2 \rangle$, $1 \le i \le N$. It follows then from what has been stated in the discussion of (32), and from (29)

$$E(\langle e_{n+1}^2 \rangle) \simeq [1 - \alpha_n \langle x^2 \rangle (2 - \alpha_n N \langle x^2 \rangle)] \cdot E(\langle e_n^2 \rangle) + 2\alpha_n \langle x^2 \rangle \langle e_{\text{opt}}^2 \rangle . \qquad (33)$$

Equation (33) is written with a time-dependent step-size parameter. It can easily be verified that

$$\alpha_{n\text{opt}} = \frac{1}{N \langle x^2 \rangle} \cdot \frac{E(\langle e_n^2 \rangle) - \langle e_{\text{opt}}^2 \rangle}{E(\langle e_n^2 \rangle)} \qquad (34)$$

leads to fastest convergence.

Usually, $E(\langle e_n^2 \rangle) \gg \langle e_{\text{opt}}^2 \rangle$ at the beginning of the equalization process. Thus we have $\alpha_{n\text{opt}} \approx 1/N \langle x^2 \rangle$ and

$$E(\langle e_{n+1}^2 \rangle) \approx (1 - 1/N)\, E(\langle e_n^2 \rangle) . \qquad (35)$$

Approximately 2.3 N iterations are then required to reduce $E(\langle e_n^2 \rangle)$ by one order of magnitude.

Since $\langle e_{\text{opt}}^2 \rangle$ is generally unknown and estimation of $E(\langle e_n^2 \rangle)$ is time-consuming, the optimum step-size parameters given by (34) cannot be realized exactly. But the optimum trajectory of $E(\langle e_n^2 \rangle)$ can be closely approached if α is controlled in the following simple manner:

a) Measure $\langle x^2 \rangle$.
b) Use $\alpha = 1/N \langle x^2 \rangle$ during the entire training period. $E(\langle e_n^2 \rangle)$ converges towards $2\langle e_{\text{opt}}^2 \rangle$.
c) Reduce the step-size parameter to $\alpha = 1/5N \langle x^2 \rangle$ when the equalizer is switched into the decision-directed mode. $E(\langle e_n^2 \rangle)$ converges further towards $1.1 \langle e_{\text{opt}}^2 \rangle$ ($\langle e_{\text{opt}}^2 \rangle + 0.5$ dB).

A step-wise reduction of the step-size parameter was already proposed by Lucky in his first paper on automatic equalization [1]. It is, however, still surprising to see how closely the optimum trajectory of $E(\langle e_n^2 \rangle)$ is approached by the simple two-step procedure suggested above. Figure 2 shows the comparison. The implementation of the procedure is indicated in Fig. 1. In practice modems are equipped with automatic gain control. If therewith $\langle x^2 \rangle$ is kept sufficiently constant, no further estimation of $\langle x^2 \rangle$ is required and the division by a variable $\langle x^2 \rangle$ in determining α is unnecessary.

The procedure proposed is also applicable when the eigenvalues of $\mathbf{R}$ are spread out over a rather wide range. This will be demonstrated in the following section by computer simulation.

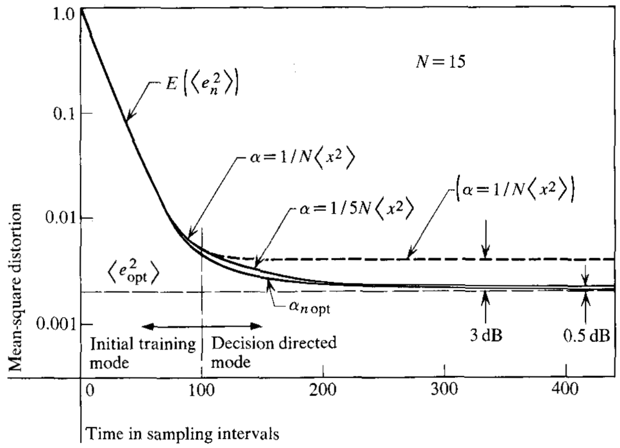
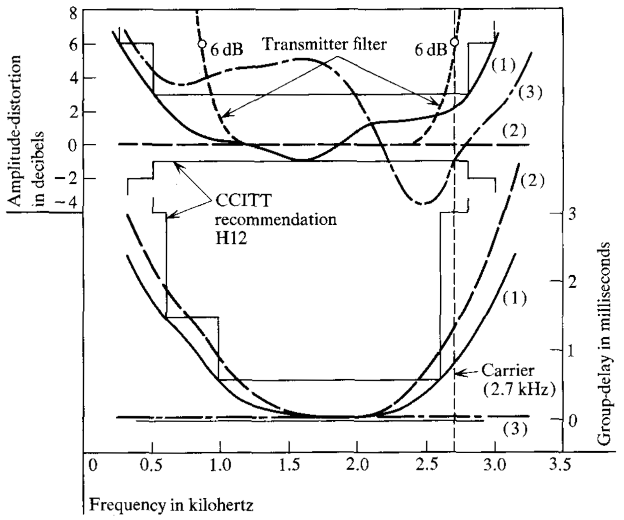


Figure 2 Speed of convergence with optimum sequence of step-size parameters ($\alpha_{n\mathrm{opt}}$) and with the step-size parameters proposed ($\alpha = 1/N\langle x^2 \rangle$ and $\alpha = 1/5N\langle x^2 \rangle$).

Figure 3 Telephone channel characteristics.



Computer simulation

Various approximations had to be made in the theoretical analysis. We shall now check the validity of the theory by comparing the theoretical results with those obtained by computer simulation. The investigation was based on the following model.

A random sequence of polar binary-signals ($a_n = \pm 1$) is transmitted over a telephone channel at the speed of 3600 baud. Vestigal-sideband amplitude modulation is used with the carrier located at 2.7 kHz. The transmitter filter exhibits symmetrical cosine-roll-off characteristics with 6-dB points at 0.9 and 2.7 kHz. Three telephone channels with characteristics shown in Fig. 3 are considered. A signal-to-noise ratio of 30 dB caused by white Gaussian noise is assumed. The equalizer comprises $N = 15$ taps. Initially, the tap gains exhibit zero values. Thus we have $\mathbf{p}_0 = -\mathbf{c}_{\mathrm{opt}}$ and $\langle e_0^2 \rangle = 1$. An ideal reference signal is assumed to be available in proper phase to the equalizer.

Two programs have been written. The first program calculates the sample values of the waveform $h(t)$ for the modulation scheme envisaged and a given telephone channel. The second program determines $\mathbf{R}$, $\mathbf{b}$, $\mathbf{c}_{\mathrm{opt}}$, the eigenvalues of $\mathbf{R}$, etc., and finds the theoretical values of $E(\langle e_n^2 \rangle)$ by iteratively applying (24). Furthermore, it generates a random data signal, adds noise to it, simulates the equalizer, and calculates $\langle e_n^2 \rangle$ at each sampling instant by evaluating (7).

At first we consider the results obtained for telephone channel-characteristic (1) (moderate amplitude and phase distortion). A step-size parameter $\alpha = 1/N\langle x^2 \rangle$ was chosen. The results of five program runs with different initializations of the random number source are presented in Fig. 4(a). Fairly good agreement of the theoretical and simulation results can be observed. On the average, however, the mean-square distortion obtained by simulation appears to converge slightly faster than is theoretically predicted. Looking for a reason, we found that this deviation can mainly be attributed to the assumption of statistical independence between $\mathbf{p}_n$ and $\mathbf{x}_n$ (equivalently $\mathbf{q}_n$ and $\mathbf{y}_n$). When additional baud intervals were introduced in the simulation between those sampling instants where tap-gain corrections are made, successive tap output signals were forced to be quasi-statistically independent of one another. In this way, without counting the additional baud intervals, a much better agreement between theory and simulation was obtained, as indicated in Fig. 4(b).

Further simulations with various step-size parameters were performed for the channel-characteristics (2) and (3) presented in Fig. 3. In order to obtain equal eigenvalues of $\mathbf{R}$ [$P^*(\omega)$ constant] with channel-characteristic (2) (phase distortion only), a transmitter filter with ideal bandpass filter characteristic had to be assumed, since otherwise aliasing would have converted phase

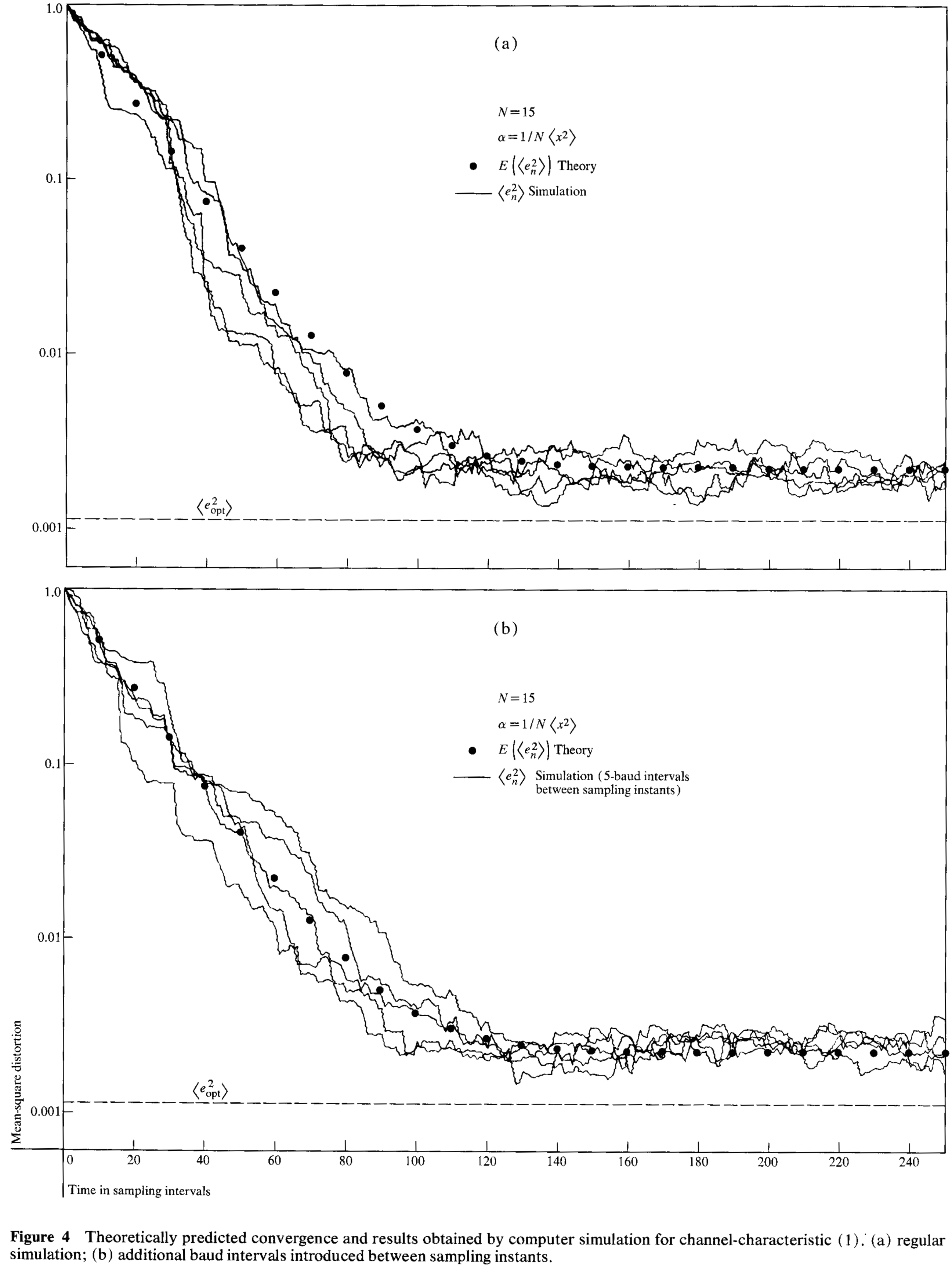


Figure 4 Theoretically predicted convergence and results obtained by computer simulation for channel-characteristic (1). (a) regular simulation; (b) additional baud intervals introduced between sampling instants.

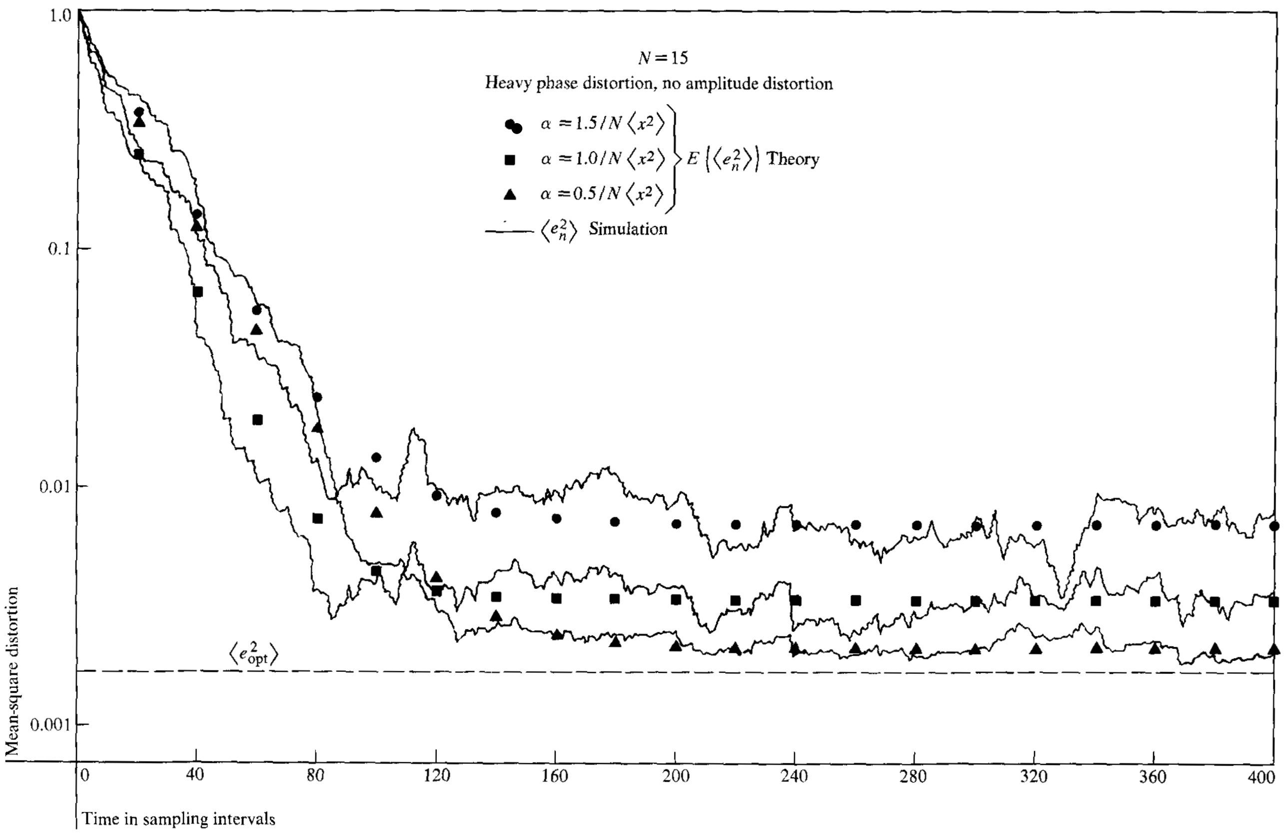


Figure 5 Theoretically predicted convergence and results obtained by computer simulation for channel-characteristic (2).

distortion into amplitude distortion. Figure 5 shows the results obtained for channel-characteristic (2). Corresponding results for channel-characteristic (3) (amplitude distortion only) are presented in Fig. 6. On the whole, the results confirm the validity of the theory, but slightly faster convergence than predicted is consistently obtained.

For channel-characteristic (3) the largest and the smallest eigenvalues of **R** differ approximately by the factor 10. This large eigenvalue spread, however, reduces the speed of convergence only by a factor of approximately 2, relative to the speed of convergence with channel-characteristic (2). Our theoretical finding that a spread of the eigenvalues of **R** affects the speed of convergence of $E(\langle e_n^2\rangle)$ less than the speed of convergence of $E(\mathbf{q}_n)$ is thus corroborated.

Figure 6 indicates that for a large spread of the eigenvalues of **R** instability occurs at a value of α smaller than $2/N\langle x^2\rangle$. In the example $\alpha = 1.5/N\langle x^2\rangle$ is close to the actual limit of stability. In this respect our theory fails for large-amplitude distortion. The discrepancy is again largely due to the assumption of statistical independence between $\mathbf{p}_n$ and $\mathbf{x}_n$.

The curves presented in Figs. 5 and 6 illustrate that in the initial phase fastest convergence is in both cases achieved by a step-size parameter close to $1/N\langle x^2\rangle$. The speed of convergence does not appear to be very sensitive to variations of α about this value. The procedure proposed at the end of the previous section for controlling α is therefore also applicable for channels that exhibit considerable amplitude distortion.

Summary and conclusion

A theory has been presented on the convergence of the expected mean-square distortion at the output of adaptive transversal equalizers that employ the well-known MS algorithm. Several approximations had to be made in the analysis, but simulation results show that quite an accurate picture of the convergence process can nevertheless be developed. The assumption of statistical independence between the tap output signals at successive sampling instants turned out to be the weakest of the approximations made.

Previous work in the field emphasized the influence of the relative difference between the largest and the smallest eigenvalue of **R** on the speed of convergence. In

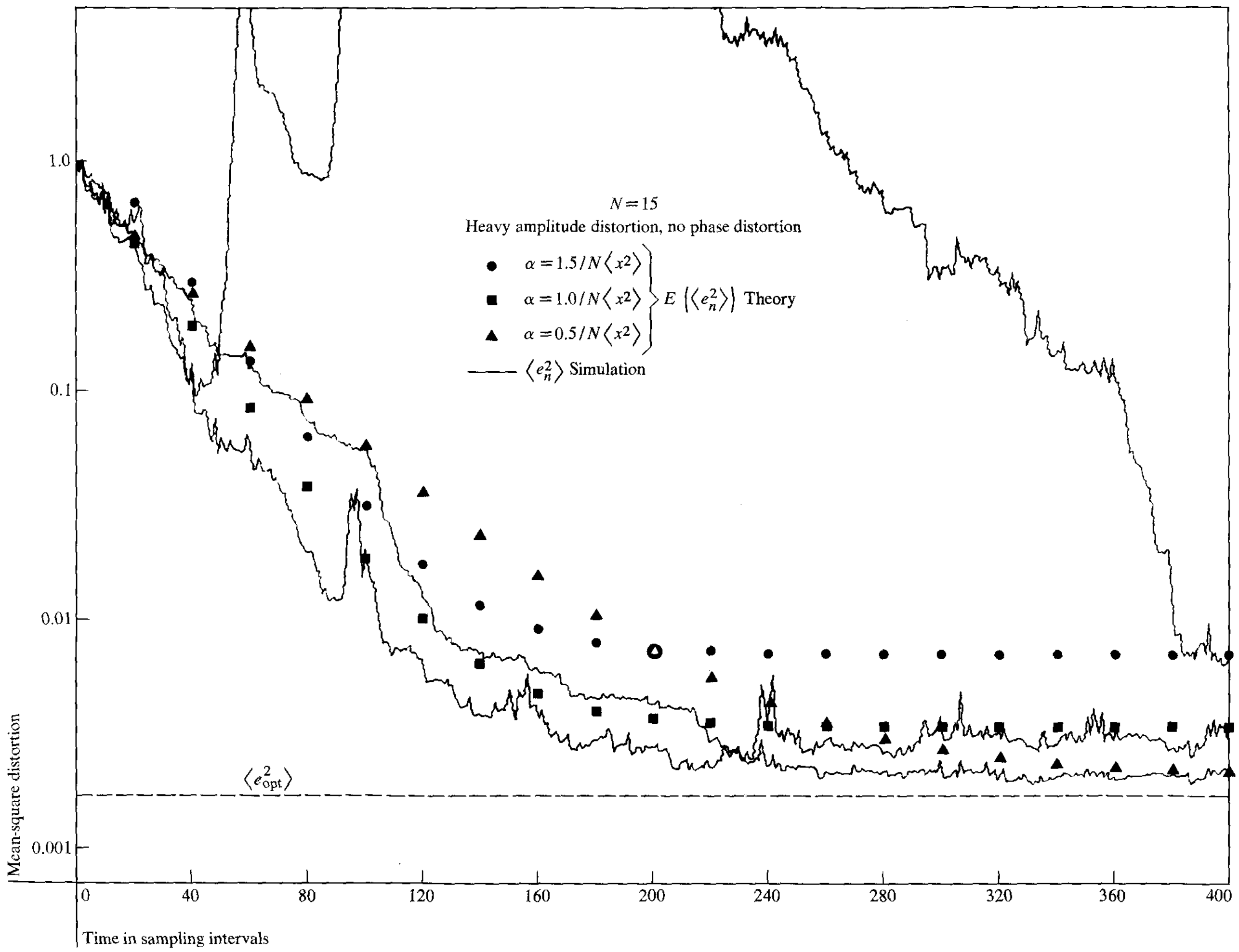


Figure 6 Theoretically predicted convergence and results obtained by computer simulation for channel-characteristic (3).

adaptive equalizers which employ the MS algorithm, these earlier results apply only to the expected tap-gain error vector, $E(\mathbf{p}_n)$. The convergence speed of the more important expected mean-square distortion, $E(\langle e_n^2\rangle)$, depends to a large extent on the number of taps: The larger N, the slower the speed of convergence. The eigenvalues of $\mathbf{R}$ have some influence on the speed of convergence of $E(\langle e_n^2\rangle)$, but this influence is not as distinct as in the case of $E(\mathbf{p}_n)$.

We have suggested a simple two-step procedure for controlling the step-size parameter in order to achieve fast convergence of the expected mean-square distortion. The true goal, however, should be to reduce as quickly as possible the expected probability of false decisions. This probability depends not only on the expectation of the mean-square distortion, but also to a certain extent on its variance. The analysis in this paper is limited in that it provides only the expectation of the mean-square distortion. It is, however, obvious that the variance of the mean-square distortion decreases monotonically with the step-size parameter. In this respect step-size parameters slightly smaller than we have proposed from the viewpoint of the expected mean-square distortion alone might be preferable.

We finally illustrate the theoretical results by a specific example. Assume an equalizer comprising 15 taps and a transmission speed of 3600 baud, as we did for the computer simulation. During the settling time the mean-square distortion should be reduced from 1 to 0.001, provided it does not level off at a larger value, i.e., $\langle e_{\text{opt}}^2\rangle > 0.001$. According to (35), with phase distortion only and the step-size parameter optimally adjusted, the equalizer settles in about 100 baud intervals, or 28 milliseconds. Moderate amplitude distortion will have no strong effect. With characteristic (1) of Fig. 3 the optimum settling time is still of the order of 30 milliseconds. For completeness, it should be noted that this does not include the additional time required for carrier acquisition, sampling clock adjustment, and synchronization with a known reference sequence.

Appendix

• *Derivation of Eq. (24)*

Analysis of the convergence properties of $s_{ni} = E(q_{ni}^2)$, $1 \le i \le N$, requires that we consider initially also the mixed quantities $E(q_{ni} \cdot q_{nk})$, $i \neq k$. Assume $\mathbf{y}_n$ and $\mathbf{q}_n$ were statistically independent of each other. From (15), observing (14), (16) and (17), it can then be shown that

$$E(q_{(n+1)i} \cdot q_{(n+1)k}) = E(q_{ni} \cdot q_{nk}) \cdot [1 - \alpha(\rho_i + \rho_k)]$$
$$+ \alpha^2 \sum_{l_1=1}^{N} \sum_{l_2=1}^{N} E(q_{nl_1} \cdot q_{nl_2}) \langle y_{ni} \cdot y_{nl_1} \cdot y_{nl_2} \cdot y_{nk} \rangle$$
$$+ 2\alpha^2 \sum_{l=1}^{N} E(q_{nl}) \langle e_{n\text{opt}} \cdot y_{ni} \cdot y_{nl} \cdot y_{nk} \rangle$$
$$+ \alpha^2 \langle e_{n\text{opt}}^2 \cdot y_{ni} \cdot y_{nk} \rangle, \quad 1 \le i, k \le N. \quad \text{(A1)}$$

According to (14) and (17) the quantities $y_{n1}, y_{n2}, \cdots, y_{nN}$, and $e_{n\text{opt}}$ are uncorrelated. Suppose they are almost statistically independent. Then

$$\langle e_{n\text{opt}} \cdot y_{ni} \cdot y_{nl} \cdot y_{nk} \rangle \approx 0.$$

Referring to (22),

$$E(q_{nl}) \approx 0, \qquad 1 \le l \le N.$$

Hence, the third line of (A1) consists of products of small quantities and can thus be neglected.

Similarly for $i \neq k$,

$$\langle y_{ni} \cdot y_{nl_1} \cdot y_{nl_2} \cdot y_{nk} \rangle \approx 0,$$
$$\langle e_{n\text{opt}}^2 \cdot y_{ni} \cdot y_{nk} \rangle \approx 0.$$

Consequently, for $i \neq k$ only the first line of (A1) is important. Convergence of $E(q_{ni} \cdot q_{nk})$ takes place if $0 < \alpha < 2/(\rho_i + \rho_k)$. We may therefore assume that these mixed terms become rapidly negligible during the equalization process if they were not already negligible from the beginning:

$$E(q_{ni} \cdot q_{nk}) \approx 0, \qquad i \neq k.$$

When this is applied to the second line of (A1), then for $l_1 \neq l_2$ we have products of two small quantities. Neglecting these products reduces the double sum to a simple summation. Considering now only the case $i = k$ we find

$$s_{(n+1)i} \simeq s_{ni}(1 - 2\alpha\rho_i) + \alpha^2 \sum_{l=1}^{N} s_{nl} \langle y_{nl}^2 \cdot y_{ni}^2 \rangle$$
$$+ \alpha^2 \langle e_{n\text{opt}}^2 \cdot y_{ni}^2 \rangle, \qquad 1 \le i \le N. \quad \text{(A2)}$$

We now approximate the higher-order expectations by second-order statistical parameters

$$\langle y_{nl}^2 \cdot y_{ni}^2 \rangle \simeq \rho_l \rho_i, \qquad 1 \le l, i \le N, \quad \text{(A3)}$$
$$\langle e_{n\text{opt}}^2 \cdot y_{ni}^2 \rangle \simeq \langle e_{\text{opt}}^2 \rangle \rho_i, \qquad 1 \le i \le N. \quad \text{(A4)}$$

With these approximations (A2) reads

$$s_{(n+1)i} \simeq s_{ni}(1 - \alpha\rho_i)^2 + \alpha^2 \sum_{\substack{l=1 \\ l \neq i}}^{N} s_{nl} \rho_l \rho_i$$
$$+ \alpha^2 \langle e_{\text{opt}}^2 \rangle \rho_i, \qquad 1 \le i \le N. \quad \text{(A5)}$$

Writing (A5) in vector form, we finally obtain (24).

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